Grounding computational cognitive models

Casimir J.H. Ludwig[1], Erik Stuchlý[2], and Gaurav Malhotra[3]
[1]University of Bristol – School of Psychological Science
[2]University of Hamburg – Institute of Psychology
[3]SUNY Albany – Department of Psychology

Abstract

Cognitive scientists and neuroscientists are increasingly deploying computational models to develop testable theories of psychological functions and make quantitative predictions about cognition, brain activity and behaviour. Computational models are used to explain target phenomena such as experimental effects, individual and/or population differences. They do so by relating these phenomena to the underlying components of the model that map onto distinct cognitive mechanisms. These components make up a "cognitive state space", where different positions correspond to different cognitive states that produce variation in behaviour. We examine the rationale and practice of such model-based inferences and argue that model-based explanations typically miss a key ingredient: they fail to explain *why* and *how* agents occupy specific positions in this space. A critical insight is that the agent's position in the state space is not fixed, but that the behaviour they produce is the result of a *trajectory.* Therefore, we discuss (i) the constraints that limit movement in the state space, (ii) the reasons for moving around at all (i.e. agents' objectives); and (iii) the information and cognitive mechanisms that guide these movements. We review existing research practices, from experimental design to the model-based analysis of data, and through simulations we demonstrate some of the inferential pitfalls that arise when we ignore these dynamics. By bringing the agent's perspective into sharp focus, we stand to gain better and more complete explanations of the variation in cognition and behaviour over time, between different environmental conditions, and between different populations or individuals.

Keywords: computational model, individual differences, model-based inference, sampling, temporal dynamics

A core aim of cognitive science is to develop testable theories of psychological functions that make quantitative predictions about behaviour. To this end, a theory may be cast as a computational model (a formal mathematical model or a computer simulation) that instantiates the psychological mechanisms and processes assumed by the theory. A computational model embodies the core assumptions of a psychological theory, along with auxiliary assumptions that are needed to connect the theory with empirical observations. Such a model may be regarded as a representation of a *target system* (Suárez & Pero, 2019), that is, the "ground truth" model that operates in an agent's brain. Many authors have written about the importance of computational modelling for theory development and testing in psychology (e.g. Farrell & Lewandowsky, 2018; Guest & Martin, 2021; Lee & Wagenmakers, 2014; Oberauer & Lewandowsky, 2019; Roberts & Pashler, 2000; Robinaugh et al., 2021; Simon, 1992; van Rooij & Baggio, 2021). To highlight just a few benefits, models allow us: (i) to generate predictions from a theory and compare different theories against data; (ii) to test "proof-of-principle" explanations of an empirical phenomenon; (iii) to identify latent psychological mechanisms and processes that underlie some cognitive capacity (e.g. decision making, object recognition, visual working memory). With advances in computing power and computing software, computational modelling is increasingly widely adopted in psychology, neuroscience and psychiatry (Huys et al., 2016; Jarecki et al., 2020; Kriegeskorte & Douglas, 2018; Robinaugh et al., 2021).

Across a wide variety of domains, researchers have taken great strides in constructing models that can explain and predict behaviour (and link this behaviour with neural mechanisms; e.g. Forstmann et al., 2011; Love, 2015; Turner et al., 2019). These models frequently take the form of *process models* that describe how information flows through the cognitive system (Jarecki et al., 2020), at what Marr (1982) dubbed the 'algorithmic level' of analysis. Crucially, most models contain parameters that allow them to be flexible (Roberts & Pashler, 2000): the same model can be used to fit data from different conditions or individuals by tuning these parameters. By fitting parameters to data, we often seek to identify "the underlying mechanism of..." some target phenomenon (e.g. a behavioural effect, cognitive capacity, neural activation). The model then provides a mechanistic explanation of this phenomenon—an explanation in terms of cognitive mechanisms and the parameters that govern their operation (Bechtel & Abrahamsen, 2005, 2010; Kaplan & Craver, 2011; Simon, 1992). This is frequently where the explanation of the target phenomenon ends: we point to a relation between the target and the estimated model parameters, and these model parameters are mapped onto meaningful psychological constructs. What more could we want?

Take, for example, evidence accumulation models of decision making (illustrated in Figure 1 and explained in more detail in the section 'Examples of model-based analysis'). These models are used widely to account for reaction time and accuracy data, and decompose these observed data into meaningful, latent psychological variables (i.e. model parameters) such as processing speed and response caution (for reviews, see e.g. Evans & Wagenmakers, 2019; Gold & Shadlen, 2001; Smith & Ratcliff, 2004). An empirical phenomenon might be that older adults are slower to respond than younger adults. Fitting the behavioural data from both groups with an evidence accumulation model suggests that older adults are slower to respond, not because they have a lower processing speed, but because their decision threshold is higher: they need more evidence to respond (e.g. Ratcliff et al., 2004). The increase in the decision threshold here is a mechanistic explanation for the difference in behaviour between young and

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Casimir J.H. Ludwig https://orcid.org/0000-0003-4656-0751 University of Bristol, School of Psychological Science, 12a Priory Road, Bristol, BS8 1TU, United Kingdom. E-mail: c.ludwig@bristol.ac.uk, web: casludwig.github.io
Erik Stuchlý https://orcid.org/0000-0002-9174-8860 Universität Hamburg, Faculty of Psychology and Human Movement Science, General Psychology, Von-Melle-Park 11, 20146 Hamburg, Germany. E-mail: erik.stuchly@uni-hamburg.de
Gaurav Malhotra https://orcid.org/0000-0002-6868-9655 SUNY Albany, Department of Psychology, Social Science 399, 1400 Washington Avenue, Albany NY 12222, United States. E-mail: gmalhotra@albany.edu
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older participants. Such model-based explanations are rife in cognitive science and we could easily have picked examples from our own back catalogue (e.g. Farrell & Lewandowsky, 2010; Ludwig, Butler et al., 2009; Ludwig, Farrell, Ellis & Gilchrist, 2009; Perez Santangelo et al., 2022).

However, such explanations are incomplete: what is critically missing is *why* and *how* parameters come to take on the values that best account for the data. In other words, the hypothesised mechanisms themselves beget explanation: Why was the decision threshold higher? What prevented older adults from adopting a lower, more appropriate threshold (given the objective of the task)? How did they come to settle on the (high) value of the decision threshold they adopted? Although such questions may be addressed in Discussion sections, modellers generally consider them beyond the scope of the model (see Starns & Ratcliff, 2010, for an attempt to address such questions for this particular case).

We recognise that the explanatory scope of models has to be restricted for them to be useful. However, why and how agents "settle" on some combination of parameters are important psychological questions in their own right that must be answered if we are to produce better and more complete explanations of cognitive, neural and behavioural phenomena. Addressing these questions involves adopting the agent's perspective and considering the constraints, information and cognitive mechanisms that are relevant while the agent attempts to navigate a "cognitive parameter space" over the course of a task. We argue that this perspective is lacking in many applications of model-based analysis, and failing to adopt this perspective has several consequences. First, models and their parameter estimates typically give a static representation of the "average latent state" that gave rise to behaviour. This average state may not be representative of the agent's state at any one point in time. Second, models are overly flexible in their predictions, because they do not adequately capture the constraints that the agent operates under. Both these factors can lead to erroneous parameter estimates and model-based inferences. Third, by focusing on this average state, modellers often overlook important and more general questions about the flexibility with which cognition and behaviour is adapted, and the mechanisms underlying this flexibility. In this article, we outline a programme of *grounding cognitive model parameters* that puts the agent's perspective centre stage, with the aim of expanding and improving explanations in the cognitive and brain sciences.

We start by examining the rationale and practice of *model-based analysis*, with examples from a variety of cognitive domains. We then operationalise the research programme by identifying three focused research questions that are typically overlooked in such analyses, but that must be addressed in order to understand why and how cognitive model parameters take on the values they do. First, we need to know what constraints are acting on the cognitive mechanisms that generated the empirical data. Second, we need to establish the agent's objectives (such as maximising speed, accuracy, reward) that drive the change in parameter values. Third, we need to work out what mechanisms and information are available to the agent to achieve their objective(s). Along the way, we also address the analytic problem of how, as cognitive scientists, we can estimate the change in parameter values over time, and the inferential pitfalls of not doing so. The overall argument we pursue here is that grounding cognitive model parameters is critical for understanding variation in cognition and behaviour over time, between different environmental conditions and between different populations or individuals. In this way, our mechanistic theories will expand their explanatory scope, and better capture the target system and its dynamics.

Model-based analysis

To help illustrate the logic of model-based analysis, we consider the broad class of evidence accumulation models of rapid decision making. We will use evidence accumulation models as a running example throughout this article, because these models are a particularly popular choice for model-based analyses. We will give a brief description of this model class (readers familiar with these models can skip the next sub-section). We then provide examples of model-based inferences, drawn from evidence accumulation models, but also from a selection of other models from other cognitive domains. These examples illustrate how models and their parameters are used to explain empirical phenomena such as experimental effects, neural signals, and individual and/or population differences. Importantly, these examples underscore that our arguments apply to cognitive models in any domain of psychological science.

Evidence accumulation models

Models of rapid choice come in many different flavours (e.g. Brown & Heathcote, 2008; LaBerge, 1962; Link & Heath, 1975; Ratcliff, 1978; Usher & McClelland, 2001; Vickers, 1970), but they all share the basic idea that evidence in favour of the decision alternatives accumulates over time to a critical threshold. There is competition in this accumulation process between the choice options and the process is corrupted by one or more sources of noise. Once the evidence for one of the alternatives reaches the critical threshold, the motor response associated with the chosen option is initiated after some "non-decision" time. The decision/drift diffusion model (DDM; Ratcliff, 1978; Ratcliff & Rouder, 1998; Ratcliff et al., 2016; Smith & Ratcliff, 2004) for two-choice tasks is probably the most frequently used model from this class. It assumes that the evidence in favour of one option is subtracted from the evidence in favour of the other option. The net evidence then drifts towards an upper or lower decision boundary, which represent the two decision alternatives.

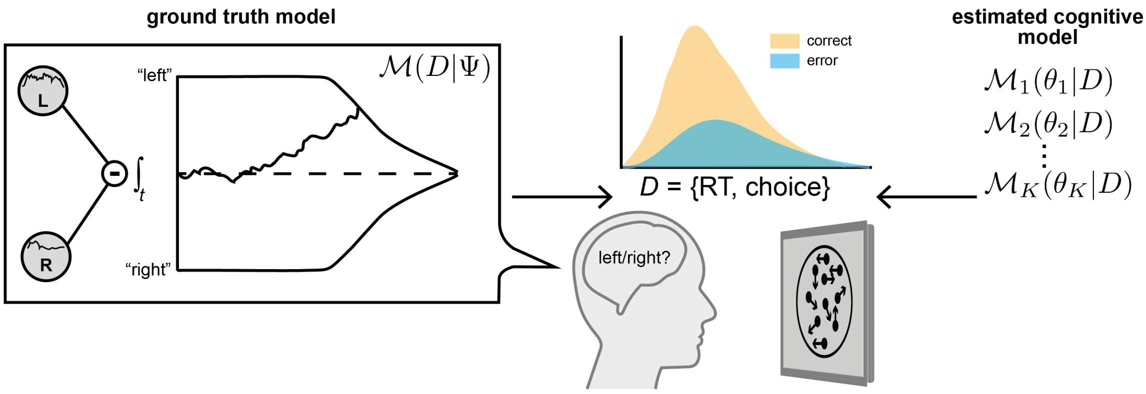
The agent illustrated in Figure 1 makes decisions in a classic motion discrimination task in line with the DDM, in this case augmented with time-varying decision boundaries (Hawkins et al., 2015; Ludwig, 2009; Smith, 2000). Suppose this model is the *ground truth.* The agent monitors the responses of sensory channels tuned to the different motion directions and subtracts the response of, say, the rightward channel from the leftward channel. Positive net evidence then points towards the motion stimulus moving left; negative net evidence points towards the motion stimulus moving right. The decision boundaries correspond to the difference in evidence that would be needed in order to commit to one or the other choice. An error occurs when, due to noise in evidence accumulation process, the integrated evidence hits the incorrect boundary (e.g. hitting the boundary for a rightward response when the pattern is moving left). Better quality sensory evidence (e.g. higher motion coherence) results in a faster drift towards the corresponding boundary. As a result, the decision is made more quickly and more accurately. The probability of making an error can be reduced by increasing the separation between the decision boundaries, but at the cost of an increase in RTs. Other parameters of the model (typically) include the mean rate of evidence accumulation, the internal noise in the evidence accumulation process, the non-decision time, the prior bias of an agent towards one decision and the inter-trial variability in the starting position and accumulation process.

Model-based inferences

Given the ground-truth model, an agent will produce observed behavioural data (in this example: RTs and choices), D, conditioned on a set of parameter values Ψ. As cognitive scientists, we want to know what $\mathcal{M}(D|\Psi)$ is. However, the true model and its parameters cannot be known. We can only try to approximate it by fitting the data with one or several models. For example, many different versions of the DDM have been proposed that include mechanisms such as: between-trial noise in the starting point; between-trial noise in the drift rate (Ratcliff & Rouder, 1998); urgency signal (Cisek et al., 2009); collapsing decision boundaries

Figure 1

Relation between the ground truth model, the data generated from the ground truth model, and the model(s) that are fit to these data.



Note. An agent performing the classic random dot motion discrimination task. They are making decisions by accumulating the net evidence in favour of one or the other motion direction towards a decision boundary. In this case, the decision boundaries "collapse" over time. This ground-truth model generates data, D, conditional on its parameter values, Ψ. The data in this example consist of RTs and choice outcomes, summarised by RT distributions for correct and error decisions. These data may be fit with a variety of cognitive models, $k = 1, \ldots, K$, each with their own set of parameters, θ_k. The challenge for the modeller is to estimate the parameters of the model, conditioned on the observed data, and to select the model that best approximates the ground-truth.

(Ditterich, 2006; Smith, 2000); attentional biases in the drift rate (Krajbich et al., 2010). This variety of mechanisms, and more generally the variety of available models within the broader class, presents the modeller with a plethora of choices when fitting the observed data (Dutilh et al., 2019). These choices give rise to the K alternative models, $\mathcal{M}_1(\theta_1|D) \ldots \mathcal{M}_k(\theta_k|D)$, on the right-hand side of Figure 1. Note the reverse dependence of the parameters on the data: our challenge is to estimate the parameter values (for each different model), given some observed data. Where several models are fit to the data, we (somehow) need to select the model that comes closest to the true model (Burnham & Anderson, 2002; Navarro, 2019; Pitt et al., 2002; Shiffrin et al., 2008).

The methods used to estimate model parameters and to select between competing models are highly active research areas in their own right, which we will not deal with in detail here (for general overviews, see Farrell & Lewandowsky, 2018; Lee & Wagenmakers, 2014). Broadly speaking, cognitive modellers seek two types of inferences: inferences through model-selection or through parameter comparison. Inferences through model-selection involve setting up models that represent competing explanations for some target phenomenon, and then selecting between these models through some procedure that trades off the quality of the fit and model complexity (e.g. various information criteria, Bayes Factors, cross-validation). Inferences through parameter comparison involve fitting the data with different sets of parameters to account for the target phenomenon. We then adopt some statistical procedure to decide which model parameters relate most strongly to this target.

For example, we may have two explanations for the slowing of reaction times with age (e.g. Ratcliff et al., 2004, as discussed in the Introduction): older people are slower to accumulate evidence (lower drift rate) or they may need more evidence before committing to a response (increased decision boundary separation). Inference through model selection would involve fitting the data from both groups with a model in which the drift rate varies between the two groups but the decision boundary remains constant, and a model in which the drift rate is constant but the boundary separation may vary. Inference through parameter comparison involves fitting the data from both groups independently with their own sets of parameters, and then performing statistical tests to assess which parameters differentiate the groups. Note that modellers often adopt both or a mixture of the model-selection and parameter comparison approaches. For instance, they may use both approaches independently as a way to provide converging evidence for a particular mechanistic signature. Alternatively, model-selection may be used to decide on a particular instantiation of the broader model class (e.g. selecting between different flavours of evidence accumulation models, such as the DDM, Linear Ballistic Accumulator, Leaky Competing Accumulator, etc.). The target phenomenon is then explained through a parameter comparison for the selected model instantiation.

In Table 1 we have compiled a (relatively arbitrary) selection of examples of model-based inferences, regardless of whether these inferences were obtained through model-selection or parameter comparisons. These examples come from a variety of models/cognitive domains: decision-making, memory, learning, (overt) attention, perception and categorisation.[1] Within each domain we list model-based explanations for different classes of target phenomena: experimental effects, individual/population differences and (where possible) neural correlates. No doubt these examples are biased by our own knowledge of the field and the reader can probably think of examples from their own area of expertise.

In each of these cases, a model-based analysis decomposes the observed data into a small number of meaningful psychological dimensions, as illustrated in Figure 2. We then look for differences along those dimensions between different experimental conditions, individuals or populations, or we try to identify neural signals that correlate with the variation along these dimensions. For instance, in Figure 2 the data consist of a sequence of reaction time and accuracy measurements. When we fit a model to these data, such as the DDM, these data are effectively projected into a low dimensional space (downward dashed arrow, estimated cognitive model). For ease of illustration, we have shown just three dimensions, corresponding to key mechanisms that may induce differences in reaction times (and accuracy) between individuals and groups (the full model has more than three parameters). Each individual may be represented as a single point within this space. For two hypothetical groups of participants (e.g. young and older people, represented by blue and orange points respectively), there is some degree of individual variation along all three dimensions. In addition, the two groups differ systematically in their position along the 'decision threshold' axis, suggesting that this mechanism specifically is responsible for group differences in the behavioural data. We can now ask why and how different people or groups occupy different locations in this space. In the remainder of this paper we break this problem down into three focused research questions that must be addressed to understand why and how cognitive model parameters take on the value they do.

[1]Readers may disagree about whether (some of) these examples really constitute *explanatory, mechanistic* cognitive models (Jarecki et al., 2020; Oberauer & Lin, 2017). However, we note that these models are often given a mechanistic interpretation or implementation (e.g. "discrete slot" versus "continuous resource" models of Visual Working Memory give rise to different flavours of mixture models; Bays et al., 2009; Van den Berg et al., 2012; W. Zhang & Luck, 2008).

Table 1

Examples of model-based analyses in multiple cognitive domains

	Target phenomenon	Model-based inference
Rapid decision making (Evidence Accumulation models)	Participants are able to trade-off speed with accuracy based on instructions.	*Decision boundaries* are higher under accuracy instructions (Voss et al., 2004).
	Older adults are slower to respond in a variety of cognitive tasks.	Older adults set *decision boundaries* too high and also have longer *non-decision times* (Ratcliff et al., 2004).
	Individual variation in responsiveness to speed-accuracy instructions.	Flexibility in *decision boundary* adjustment is linked to connectivity between pre-supplementary motor area and striatum (Forstmann et al., 2010).
Episodic memory (Multinomial Processing Tree models)	Better source memory for information that is consistent with prior expectations.	*Guessing parameter* is biased by expectations about the source derived from schematic knowledge (Bayen et al., 2000).
	Memory deficits in Alzheimer's Disease.	Greater deterioration in *immediate retrieval primacy* compared to *immediate retrieval recency* with disease progression (Lee et al., 2020).
	Activity in Posterior Parietal Cortex (PPC) is correlated with item and source memory.	PPC activity linked to item and source memory *guessing biases* (Pergolizzi & Chua, 2016).
Visual Working Memory (Mixture models)	Visual Working Memory (VWM) performance decreases with set size.	The decrease in performance with memory load largely reflects a decrease of the *probability that the item is represented in memory* (W. Zhang & Luck, 2008).
	VWM deteriorates with age.	The *precision of VWM representations* decreases with age; the *probability of reporting a non-target feature* increases with age (Peich et al., 2013).
	VWM load and delay related activity in a network of frontal, parietal and occipital regions.	Load-dependent increases in *precision variability* is linked to the quality of neural representations in the superior Intraparietal Sulcus (Galeano Weber et al., 2016).

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Table 1

Examples of model-based analyses in multiple cognitive domains – continued

	Empirical finding	Model-based inference
Learning (Reinforcement Learning models)	"Positivity bias" in learning: people often learn more from positive compared to negative prediction errors.	*Learning rates* depend on outcome valence, but in opposite ways for factual and counter-factual learning (Palminteri et al., 2017).
	Impaired performance in reversal learning in participants with more Obsessive Compulsive Disorder symptoms.	Higher obsessive-compulsive symptoms linked to increased (subjective) *transition uncertainty* (Fradkin et al., 2020).
	Exploration-exploitation dilemma in reward-based learning.	*Inverse temperature* is related to Locus Coeruleus - Norepinephrine system, suggesting this system is involved in controlling choice strategy (Jepma & Nieuwenhuis, 2011).
Active vision (Dynamic eye movement control models)	Oculomotor control is modulated by reading text in different lay-outs (e.g. normal vs inverted), suggesting cognitive control over eye movements.	Text lay-out manipulations affect the *perceptual span* and the *scale of an autonomous timer* mechanism (among others; Rabe et al., 2021).
	Individual differences in scan paths during scene viewing.	Differences in *Attentional window size* parameters (among others) mediate variation in saccade amplitudes between individuals (and tasks) (Schwetlick et al., 2023).
	Development of reading ability manifests itself in changes in fixation duration and saccade amplitudes.	Primary difference between children and adults lies in the *rate of lexical processing* (Reichle et al., 2013).
Perceptual learning (Template matching models)	Orientation discrimination in noise improves with training (thresholds decrease over time).	Learning is mediated by narrowing of *filter tuning* (exclusion of external noise) and suppression of *additive internal noise* (Dosher & Lu, 1999).
	Deficits in (spatial) vision in adults with amblyopia.	Visual performance deficits in amblyopia are mediated by increased *internal noise* and deficient *perceptual templates* (Xu et al., 2006).
	Perceptual learning may be mediated by changes in early sensory encoding or by the changes in the read-out of the early sensory code.	Learning results in changes in the *perceptual weighting* of the input and is reflected in anterior cingulate cortex activity, suggesting a higher-order, non-sensory locus of perceptual learning (Kahnt et al., 2011).

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Table 1

Examples of model-based analyses in multiple cognitive domains – continued

	Empirical finding	Model-based inference
Categorisation (Category learning models)	Category learning is more impaired by a concurrent task when learning simple, explicit rules compared to more complex implicit rules that require integration across multiple dimensions.	A deficit in learning explicit rules is best captured by a decline in the *ability to select new rules*; this decline leaves implicit rule learning unaffected (Waldron & Ashby, 2001).
	Intact category learning despite impaired recognition for individual items in amnesic patients.	The *threshold for forming a new representational cluster* is different between amnesic patients and controls (Love & Gureckis, 2007).
	Prototype and exemplar models generate similar predictions for behavioural categorisation performance.	Trial-by-trial activation of lateral occipital cortex and posterior parietal cortex correlates with a measure of *representational match* derived from exemplar models (Mack et al., 2013).

Note. The examples were chosen to span a range of model classes, cognitive capacities and empirical phenomena. In particular, for each domain/model class, we selected a basic experimental effect, a population or individual difference, and (where possible) a neural correlate. We have focused generally on relatively well-established models and generally highly cited papers. Verbal labels for model parameters (or model-based metrics) are marked in italics.

What are the constraints on the model parameters?

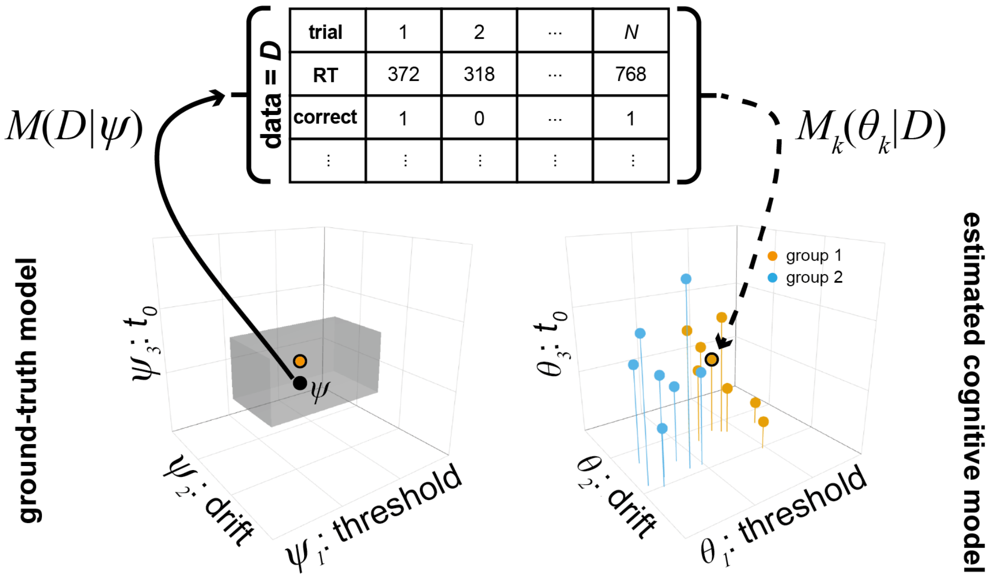
Consider again the agent illustrated in Figure 1. They perform a perceptual decision making task using one particular instantiation of the broad class of evidence accumulation models. This system is characterised by a set of parameters, Ψ. The true values for Ψ may be represented as a point in a *cognitive parameter or state space*, where the dimensions of the space represent the parameters of the ground-truth model. Figure 2 (bottom left, ground-truth model) illustrates the agent's position in a three-dimensional (sub-)space. The data generated by the agent depends on their (ground-truth) position in this space (upward solid arrow).

It may be tempting to explain why an agent occupies a particular position in this space by identifying why this position is special in some way. For example, we could justify an observed value of decision threshold by comparing it to the value that maximises some objective in an experiment (e.g. Balci et al., 2011; Bogacz et al., 2006; Hawkins et al., 2012; Starns & Ratcliff, 2010; Zacksenhouse et al., 2010). Indeed, we discuss the importance of objectives, and individual variation therein, in the next section ('What is the agent's objective?'). For now, suffice to say that agents may attempt to achieve some objective by moving about strategically in the cognitive parameter space.

Almost all cognitive models allow for some degree of strategic control over (a subset of) model parameters. Take each of the model classes represented in Table 1 in turn. *Evidence accumulation models* assume that agents can trade-off accuracy against speed by controlling the decision threshold (Ratcliff & Rouder, 1998; Voss et al., 2004). Parameters such as the drift rate and non-decision time are under much less strategic control, but may still be modulated to some extent through, for example, selective attention and arousal. *Multinomial Processing*

Figure 2

Cognitive modelling involves a mapping between high-dimensional data and a low dimensional, psychologically meaningful parameter space.



Note. The matrix at the top shows some hypothetical data for a given participant, which consists of a sequence of reaction times and accuracy measurements, but could include other dependent variables (e.g. skin conductance, EEG power in a particular spectral band). These data are generated by a ground-truth model (bottom-left) of the type shown in Figure 1. For the sake of illustration, we only show a 3-dimensional sub-space of parameters: the height of the decision threshold, the mean drift rate and the non-decision time, t_0. Suppose these ground-truth model parameters are subject to constraints that limit the room for manoeuvre in the cognitive parameter space. These constraints are represented schematically by the shaded cuboid. An agent occupies a point in this space, which results in a noisy sample of observed data (upward solid arrow from the black point; $\mathcal{M}(D|\Psi)$). The modeller takes the observed data and fits one or more models to these data by tuning the free parameters (downward dashed arrow; $\mathcal{M}_k(\theta_k|D)$). The data from a participant (N tuples of the dependent variables) are then mapped to a point in this low dimensional space (estimated cognitive model, bottom right). Different participants will occupy different positions in this space and, in this example, there is also a systematic difference between two groups of participants along the decision threshold axis. The participant who generated the data is highlighted with a black outline (and re-plotted in the ground-truth model). Note that in this illustration we assume that the estimated cognitive model is a good approximation of the ground-truth model; that is, the two models share these underlying dimensions. Nevertheless, the agent's position in the cognitive parameter space is not recovered exactly in the parameter estimates: the estimated parameter vector (orange) is not aligned with the parameters Ψ in the ground-truth model that generated the data. There may be a number of reasons for this discrepancy, such as: (i) the structure of the estimated cognitive model may not align exactly with that of the ground-truth model (i.e. the two models may not have exactly the same dimensions of variation); (ii) even if the estimated cognitive model matches the (unknown) ground-truth model, the data may not be sufficiently diagnostic for identifying all the parameters of the ground-truth model accurately (e.g. due to methodological limitations, and the data being a noisy and finite sample from the ground-truth model); (iii) there may be error in the estimation procedure and/or parameter trade-offs.

Tree models of memory typically contain at least one or two thresholds for carving up the decision space (Batchelder & Riefer, 1990; Bayen et al., 1996) and these thresholds are to some degree under strategic control (e.g. influenced by expectations in Bayen et al., 2000). *Flexible resource models of visual working memory* (e.g. Bays & Husain, 2008) assume that there is a limited capacity resource that may be configured adaptively according to the demands of the task, which in turn determines the precision with which items are represented. *Reinforcement learning models* feature learning rates that may be adjusted to alter the sensitivity to recent reward feedback and volatility in the environment (Behrens et al., 2007), as well as parameters that control the degree of exploration (Daw et al., 2006; Wilson et al., 2014). *Models of eye movement control assume* that observers have some control over their perceptual span (the region from which perceptual information is taken to select the next fixation location) and the rate of an internal timer that paces the eye movements (Engbert et al., 2005; Nuthmann et al., 2010). *Perceptual template matching models* typically assume that observers can tune the properties of their perceptual template, such as its shape and its sensitivity (Lu & Dosher, 2008). *Models of category learning* may include a threshold for the abstraction of exemplars to a new category representation (e.g. cluster, prototype or rule) (Love & Gureckis, 2007).

However, before we attribute an agent's position in the cognitive parameter space to strategic control, we should consider the various constraints that limit the space that is accessible to the agent. That is, the more constrained the range for a particular parameter is, the less likely it is that the participant settled on a particular value through strategic control. Rather, in that case, the explanation for the agent's position lies in these constraints and they should form part of the model. In this section, we highlight three broad classes of constraints—biophysical, environmental and cognitive—and then discuss how such constraints may be included in our models.

Constraints on strategic control

Biophysical constraints. Some parameters will be subject to "hard-wired" constraints imposed by physical or physiological limits. For instance, afferent and efferent delays (making up non-decision time in models of evidence accumulation) cannot be reduced below a certain minimum time that is determined by the speed of communication along the neural pathways for sensory signalling and movement production (e.g. Bullier, 2001; Munoz & Wurtz, 1995; Pruszynski et al., 2010). Metabolic costs associated with neural firing and synaptic transmission may limit representational capacity (Attwell & Laughlin, 2001; Levy & Baxter, 1996). Indeed, such costs may contribute to the spatial and/or temporal filtering characteristics of the sensory apparatus, which constrain the fidelity with which an input may be represented (Vincent & Baddeley, 2003). Moreover, noise is inherent in neural processing and is often correlated between neurons, so that it cannot be eliminated completely by spatial or temporal pooling (Averbeck et al., 2006; Shadlen et al., 1996). However, in our modelling we often ignore these factors. That is, non-decision times are often left to vary over a wide range, sometimes resulting in implausibly large estimates (e.g. over half the reaction time in Ludwig, 2009). Likewise, the mechanisms that encode the inputs are typically left unspecified and the noise parameters in the model are often used as "scaling" parameters (i.e. fixed to some arbitrary constant to make the model identifiable), or otherwise simply tuned to account for the variability observed in the data.

Environmental constraints. Properties of the environment may directly constrain some of the model parameters to be within a certain range. One such property is the information available to perform a particular task. Information is often incomplete and uncertain (at least in "large-world" problems; Gigerenzer & Gaissmaier, 2011; Simon, 1955) and, even if all the relevant information is available, there may not be enough time to use all that information. Even in simple experiments, information is often noisy and feedback about performance is often either withheld or stochastic. For example, in the classic perceptual decision making task illustrated in Figure 1, the stimulus itself is noisy (and sometimes non-stationary; e.g. Holmes et al., 2016; Ludwig & Evens, 2017). The quality of this information affects the rate at which evidence is accumulated: the drift rate for a low coherence random dot motion stimulus is lower than that for a high coherence motion pattern. However, other than constraining the model to respect this ordinal relation, drift rate parameters are typically left free to vary to accommodate the variation in behaviour introduced by the variation in the quality of information.

Cognitive constraints. The computations that can be performed on the available information in the available time may be limited further by constraints on cognitive capacities (e.g. processing speed, attention, working memory capacity, etc.) and costs associated with using these capacities (Anderson, 1990; Howes et al., 2009; Lieder & Griffiths, 2020; Simon, 1976). For example, in evidence accumulation models of multi-attribute choice, attention cannot be deployed to all different attributes of the different choice options simultaneously (Busemeyer et al., 2019; Busemeyer & Townsend, 1993). Rather, comparisons are made on an attribute-wise basis with attention switching sequentially between attributes and options. This pattern of attending to different attributes at different points in time, produces temporal fluctuation in the drift rate during a choice epoch, even when the environment (i.e. information about the choice options) remains constant. Although we may equip a model with such attention switching mechanisms (Busemeyer & Townsend, 1993; Krajbich et al., 2010), the drift rates associated with different attributes and options, and the temporal profile of attention switches are left free to vary over a wide range in order to account for the observed behaviour.

Identifying and modelling constraints

A useful way to think about these different constraints is that they limit the room for manoeuvre in the cognitive parameter space. It is likely that along any one dimension, a (weighted) mixture of constraints limits the range of parameter values that may be adopted (and/or the resolution with which parameters can be varied). Moreover, the range will vary between parameters: some parameters will be constrained more strongly than others. In Figure 2 (ground-truth model, bottom left), the grey shaded cuboid represents schematically the constraints that limit the possible positions that an agent may adopt. This volume represents the part of the cognitive parameter space that is accessible to the agent: this is the space in which the agent can exert strategic control over the parameters.

When we fit a model to data, it would clearly be helpful to incorporate these constraints in our parameter estimates. Consider what would happen if we ignored the constraints in Figure 2. That is, from the perspective of the modeller, the shaded cuboid does not exist. This absence renders the space of possible parameter estimates wide open, making the model more flexible than it should be. That is, the parameters are free to vary over too wide a range, producing greater freedom in the model predictions (Roberts & Pashler, 2000). As a result, there is a good chance that our parameter estimates (such as those shown for the estimated cognitive model) fall outside the plausible region given by the properties of the environment, and the cognitive and neural systems involved in the task (e.g. implausibly large estimates of non-decision time in Ludwig, 2009). For example, the estimated parameters shown in Figure 2 (bottom right), display greater variation along the non-decision time axis than the constraints should allow for. Such erroneous estimates will have knock-on effects on other parameter estimates: too long a non-decision time implies too short a decision time, which can only be produced by increasing the mean drift rate and/or lowering the decision threshold.

Of course, often modellers *do* impose constraints on parameters when fitting models, for instance by imposing hierarchical structure in models and priors on the parameters (Lee &

Wagenmakers, 2014). However, common practice is to leave such priors relatively vague and letting the data "speak for themselves". Even if priors are more informed, they are typically determined by the modeller's experience with what typical parameter estimates look like for the model in question, rather than mechanistic considerations. For the purpose of grounding cognitive parameters, constraints on parameters should be informed by such mechanistic considerations.

For example, the drift rate is thought to be determined primarily by the input (i.e. environmental constraints) and the transduction of that input into an internal representation (biophysical constraints). Therefore, if we know the nature of the inputs an agent encounters and have a model of the basic (sensory) machinery by which those inputs are represented, we can estimate the drift rate for any one specific input. At least for perceptual tasks, we can use our knowledge of early visual coding in order to turn an input image (sequence) into a time-varying drift rate (Ludwig, 2009; Smith, 1995; Zylberberg et al., 2012). An exciting, recent development is the emergence of Convolutional Neural Networks that can be deployed as front-ends to decision-making systems, in order to generate internal visual representations of more complex inputs (e.g. medical images in Holmes et al., 2020). Augmenting the basic evidence accumulation model with a perceptual front-end obviously introduces additional complexity to the model. However, basic perceptual mechanisms have been well characterised psychophysically and/or physiologically, and deep neural networks are typically trained independently on a different task and different data sets. To the extent that this front-end can be specified independently, these additional mechanisms can actually *reduce* the degrees of freedom of the model, by removing the freedom for the drift rate to take on any value. Note that this approach is not limited to perceptual tasks or, indeed, evidence accumulation models (see Sanders & Nosofsky, 2020; Zou & Bhatia, 2021, for examples of this approach in category learning with naturalistic visual or linguistic inputs, respectively).

To summarise, cognition and behaviour are dependent on the position of an agent in a cognitive parameter space. This state space is formed by the dimensions (i.e. parameters) of the ground-truth model that the agent brings to bear on a particular task. In this section, we have reviewed some of the different sources of constraints that limit the room for manoeuvre in this multidimensional space. A (weighted) mixture of constraints will act on each dimension, and this weighted mixture may vary for the different dimensions. As a result, movement along some dimensions will be more constrained than movement along other dimensions. This variation in freedom of movement along different dimensions corresponds to variation in the degree to which parameters may be controlled strategically.[2] Grounding cognitive parameters involves identifying and, if possible, modelling these constraints. When these constraints are not adequately incorporated, models are too flexible in that they assume people can "go anywhere" in the parameter space. As a result, we are likely to make incorrect inferences about the agent's position and movement in the state space. Of course, a key question is why an agent needs to move in this space at all: why are they exerting strategic control? In order to address this question, we need to understand what it is that they are trying to achieve, that is, their objective.

[2] In this article, we often refer to "strategic control parameters", which gives the impression that some parameters are under strategic control (e.g. decision threshold) and others are not (e.g. non-decision time). This terminology is merely useful shorthand and, as illustrated in Figure 2, this dichotomy is false. Exerting strategic control is a matter of moving about in a multi-dimensional space. It is possible that an agent simplifies the problem by ignoring those dimensions that are under less control (thereby reducing the dimensionality of the space they have to navigate). Nevertheless, even then the remaining parameters may vary in the degree to which they can be controlled strategically.

What is the agent's objective?

A satisfying answer to the *why* question—why does an agent adopt a particular position in the cognitive parameter space in Figure 2—would be a *rational explanation* of this position (Anderson, 1990; Lieder & Griffiths, 2020; Oaksford & Chater, 2007). Is the position instrumental in achieving some objective? Different positions in this space will generate different behaviours and consequently lead to different outcomes. The utility of a particular outcome will depend on an agent's objective. In a rational analysis a single, normative objective is typically assumed. The question is then how the agent might achieve this objective. Unfortunately, many (if not most) psychological experiments are not designed with the goal of inferring a participant's objective and linking it to the estimated model parameters (see below for examples of exceptions). As a result, objectives are free to vary and there may be a great deal of variability between participants and populations in their chosen objectives. This variability in objectives is a source of individual or population differences in cognition and behaviour. In this section, we will discuss the importance of setting a clear task objective, but also the importance of recognising individual differences in the objectives that are *actually* adopted by agents.

Variable objectives = variable behaviour

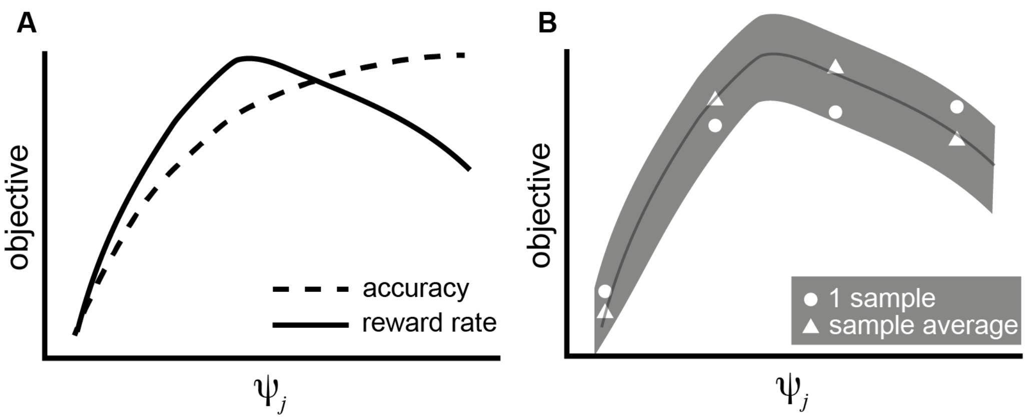
Figure 3A illustrates schematically the relation between a strategic control parameter Ψ_j and two different objectives. For example, the objectives might be reward rate and accuracy, with the decision threshold as the control parameter. The relation between the height of the decision threshold and reward rate is non-monotonic: set the decision threshold too low and the agent will make too many errors; set the decision threshold too high and they will spend too long on any one decision when they should be moving on to the next trial (i.e. reward opportunity). In contrast, the relation between decision threshold and accuracy is monotonic, with accuracy improving up to a ceiling as the decision threshold is increased. Maximising the objective involves varying the relevant strategic control parameter(s) and finding the peak of the function (we discuss this search process in more detail in 'What mechanisms and information are available to navigate the cognitive parameter space?'). As such, different (groups of) participants who have adopted different objectives are likely to settle on different values of this control parameter (regardless of whether they succeed in finding the optimal parameter value).

In most cognitive modelling endeavours the participant's objective is not linked to the parameter estimates and, in any event, the objective in many psychological experiments is either vaguely specified or not at all. Take, for example, the common instruction to "try to respond as quickly and as accurately as possible". It is left for participants to figure out what these instructions mean for them. Some participants may only want to achieve a minimum acceptable level of accuracy and want to leave the lab as quickly as possible (Hawkins et al., 2012). These participants may adopt a low decision threshold so that they respond quickly, but at a cost of a higher error rate. Others may care more about accuracy than response speed and set their threshold higher (e.g. Bohil & Maddox, 2003; Starns & Ratcliff, 2010). Still others may be aiming for a certain level of confidence in their decisions (Vickers & Lee, 1998) or the maximum reward rate (Bogacz et al., 2006). If participants are left to make up their own objectives, the variation therein will be a source of uncontrolled variance in the data and in any parameter estimates derived from quantitative model fits to those data.

An obvious solution to counteract this uncontrolled variance is to formulate a *task* objective explicitly, for instance by specifying a clear and fully transparent incentive structure. This approach is standard in experimental economics (for reviews, see Camerer & Hogarth, 1999; Hertwig & Ortmann, 2001; Houser & McCabe, 2014), and has found some traction in psychology as well. For example, Malhotra et al. (2017) designed a decision-making experiment

Figure 3

Objectives as a function of a single strategic control parameter.



Note. A–Different objectives may depend on the value of a strategic control parameter in different ways. The solid line shows a non-monotonic relation between an objective (e.g. reward rate) and ground-truth parameter, Ψ_j (e.g. decision threshold). The dashed line shows a different objective (e.g. accuracy). Agents looking to maximise these different objectives will settle on different values of Ψ_j. For the sake of illustration, the curves are drawn schematically and scaled arbitrarily. B–The agent's estimates of the objective will often be uncertain, illustrated with the error ribbon around the reward rate objective from panel A. The agent samples the objective at four different values of control parameter Ψ_j. The circles illustrate the noisy objective estimates that might result from taking just a single sample. The triangles illustrate the objective estimates that result from averaging over multiple samples taken at each point in the parameter space. Clearly, integrating the objective estimates over multiple samples results in more accurate estimates, but at a greater temporal cost. The thin black line shows the true (mean) objective.

as a game where participants had a limited time to collect as many reward points as possible and had to make correct decisions to obtain a reward. Therefore, participants were incentivised to balance their speed and accuracy in order to maximise their reward rate (see also Evans et al., 2017; Starns & Ratcliff, 2012). In un-speeded decision paradigms, participants may be incentivised to maximise their accuracy and confidence judgements jointly, through the use of Brier scores (or variations thereof; cf. Bang et al., 2017; Brier, 1950; Yu et al., 2015). In experimental economics, even when incentives do not shift behaviour on average (compared to some baseline that involves no or a different kind or level of incentives), they tend to make behaviour less variable (Camerer & Hogarth, 1999). Presumably, this reduction in variability is at least partly down to more people adopting the same, experimenter-intended objective, and having (learnt) some knowledge of how to aim for that objective given the task environment.

In addition to specifying an explicit objective, participants also need knowledge of how this objective relates to the task environment, such as the number of trials, the penalties associated with incorrect decisions, dependence on other agents' actions, and so forth. These task features may be conveyed through instructions and/or through experience with the task (learning). More generally, people may vary in how they represent the task environment due to variations in background knowledge, cognitive capacities, and how they perceive and respond to feedback from the environment (Szollosi et al., 2023). To the extent to which this variability is left unconstrained, it will result in variation in cognition and behaviour.

When a clear task objective has been defined, and the task environment is structured so as to encourage maximising this objective, agents may nevertheless fail to meet the objective. This failure may have various reasons, including limitations on cognitive and/or temporal resources (Anderson, 1990; Howes et al., 2009; Lieder & Griffiths, 2020; Simon, 1976). If agents replace the specified task objective with their own personal objective, what appears to be sub-optimal performance according to the standards defined by the task, may actually be adaptive with regard to the agent's personal objective(s) (Rahnev & Denison, 2018). For instance, in reward rate experiments with blocks of a fixed duration, participants may favour accuracy over reward rate (Balci et al., 2011), perhaps because it is easier to track performance accuracy than it is to estimate reward rate. As a result, they end up setting their decision thresholds too high with respect to the defined task objective (e.g. Balci et al., 2011; Bogacz et al., 2010; Bohil & Maddox, 2003; Starns & Ratcliff, 2010). Alternatively, agents may adopt *robust* strategies that incorporate their uncertainty about the variables that determine the task objective (e.g. maximise the minimum reward rate, given uncertainty about inter-trial intervals; Zacksenhouse et al., 2010). Therefore, the *actual*, variable objectives adopted by participants, and their relation to estimated model parameters, should be investigated.

Investigating variable objectives

We consider three possible approaches to investigate such individual (or population) differences in objectives: (i) assess which objective function best explains behaviour; (ii) elicit subjective reports; (iii) take independent measurements of task-related variables. First, we suggest that researchers run model comparisons to assess how different objective functions relate to the strategic control parameters under investigation. When two populations or individuals differ in estimated parameter values (e.g. as illustrated in Figure 2), this analysis can help identify whether a difference in objectives can account for observed variability in parameter estimates. For example, in experiments where participants are asked to maximise reward rate, we might find that a sub-set of participants are unable to find the optimal value for the decision threshold and set their threshold too high (Bogacz et al., 2010; Starns & Ratcliff, 2012; Zacksenhouse et al., 2010). We may then assess whether the data from these "sub-optimal" participants are better modelled by assuming that they were aiming for a different objective, such as maximising accuracy (cf. Figure 3A; Balci et al., 2011), minimising time spent in the lab without making too many errors (Hawkins et al., 2012), or maximising minimum reward rate (Zacksenhouse et al., 2010). This approach may be generalised to settings where there is no specific task objective, in which case the variability in adopted objectives is likely to be greater.

Our second suggestion is to investigate objectives more directly. For instance, where possible, we can ask participants to report their objectives or to elicit ratings of how much they care about different possible objectives. Even in simple choice experiments of the type discussed throughout this paper, we might elicit ratings of how much participants cared about speed, accuracy, reward and so forth. Where objectives are less accessible for subjective report, we may be able to infer them by giving participants a choice between different tasks that embody different objectives. For example, Kool and Botvinick (2014) let participants choose (and switch) between a high-effort, high-reward task and a low-effort, low reward task. Presumably, participants who spent more time pursuing the high-effort task cared more about the reward than participants who chose the low-effort task, who likely preferred to save energy. It is also plausible that this type of variability is linked to stable personality traits such as 'need for cognition' (Cacioppo et al., 1996; Gheza et al., 2023; Sandra & Otto, 2018). Either way, a critical step is to relate the individual differences in subjective reports or inferred goals to the variability in estimated parameters.

Finally, differences in estimated parameters between participants may not stem from a difference in their objectives, but from their ability to estimate these objectives. Therefore, our third suggestion is to take independent measurements of task-related capacities that are involved in estimating the objective. For example, for participants to estimate an objective like reward rate, they need to be able to keep track of the amount of time that has elapsed (Zacksenhouse et al., 2010). Measuring individual differences in timing ability may then offer insights into why participants do not adopt optimal decision thresholds (Zacksenhouse et al., 2010) or collapsing boundaries (Miletić & van Maanen, 2019). Again, the variability in these task-related variables needs to be linked to the variability in the estimated parameters. We have not encountered many examples of this last, critical step and we believe this is fertile territory for a programme of grounding cognitive model parameters.

In summary, task objectives are not fully specified in most psychological studies and participants can bring their own objectives to bear on the task. This variation in objectives can lead people to different positions in the cognitive parameter space, resulting in variation in behaviour. Understanding these individual and/or population differences is key to understanding why cognitive model parameters take on the values they do. However, even when different agents have adopted the same objective, they may still end up in different locations. To understand this source of variation, we need to examine the information and cognitive mechanisms available to agents to navigate the cognitive parameter space. Therefore, we now take a closer look at the dynamics of strategic control.

What mechanisms and information are available to navigate the cognitive parameter space?

Suppose a participant has adopted a certain objective—either the one prescribed by the experimenter or one they have defined for themselves. They now have to try to achieve this objective by varying the strategic control parameters of the ground-truth model. Understanding *how* they vary these control parameters is a critical ingredient of a more complete explanation of the participant's behaviour. In essence, this is "simply" a standard optimisation problem: the agent has to move around in the cognitive parameter space in order to find the peak of the objective function.[3] There are many algorithms available to solve such problems, such as gradient descent, the Nelder-Mead simplex method, genetic algorithms, simulated annealing and so forth. As such, a natural hypothesis is that human participants try to approximate this kind of algorithm (Busemeyer & Myung, 1992), either during performance of some specific task or over the course of their lifetime (or, indeed, across generations over the course of evolution). Although few papers make such a claim explicitly, the assumption that participants perform some kind of optimisation over the short timescales of a specific task is often implicit (including in our own work; e.g. Malhotra et al., 2017). In this section, we examine this idea critically. In particular, we highlight how this problem is much more complex for an agent than it appears at first sight, due to limitations in the cognitive resources and information available for a task. This complexity is likely to bear on the mechanism(s) adopted by agents to navigate the parameter space in many realistic environments.

Uncertainty in objective estimates

To appreciate the difficulty of the problem faced by the agent, consider again the simple, one-dimensional objective function from Figure 3A. This objective function is set by the environment (or the experimenter in a cognitive task) and the agent is unlikely to have much

[3] This function is also often referred to as a 'cost' or 'loss' function (e.g. in statistics or economics) or 'fitness' function (e.g. in theoretical biology). Note that where the objective is framed as a cost or loss, the function will need to be minimised, but flipping of the objective makes no difference to our arguments.

prior knowledge of global shape of the objective function that they can use to guide their task performance. The best they can probably do is to *sample* the objective at various different points along the strategic control parameter axis (x-axis in Figure 3A) and experience *local* feedback about the objective value at those points. In the absence of prior knowledge or a rich internal model of the global function, the agent cannot sample the objective through mental simulation, but actually has to interact with the environment. That is, for any given value of the control parameter, the agent generates an action, observes the outcome and updates their estimate of the objective function at that location. They then choose another point along the axis, observe another outcome and form another local estimate. The goal of the agent is to estimate simultaneously the underlying objective function and to move to a point along this axis that maximises this objective function.

In many realistic situations, estimating even the local value of the objective function is not straightforward because information gained about the objective is likely to be uncertain (Mikhael & Bogacz, 2016). We distinguish between two sources of uncertainty. First, the relationship between actions and their outcomes is likely to be stochastic. For example, a trader may adapt their weighting of the different cues that they use to predict the performance of a company and make investment decisions. However, whether they actually get a return on an investment (and the size of that return) is subject to various factors beyond their control. We refer to this source as *environmental uncertainty.* Second, even when the environment provides deterministic feedback about action outcomes, our perception and/or interpretation of that feedback may be error-prone, perhaps due to internal noise in the sensory and cognitive mechanisms used to process the feedback. Returning to the trader, some investments provide more predictable outcomes, but the utility of those outcomes may vary depending on the internal state of the investor, recent performance, the performance of other funds in their portfolio and so forth. We refer to this source as *representational uncertainty.* Clearly, both environmental and representational uncertainty compound each other in environments in which feedback is stochastic. Therefore, in what follows we assume that estimates of the objective function are uncertain, without specifying the source of that uncertainty.

We illustrate this uncertainty in our estimates of the objective by the grey ribbon around the objective in Figure 3B. The ribbon shows the scale of a distribution from which the objective estimates are sampled when varying strategic control parameter Ψ_j. Suppose the agent samples the objective at four parameter values. For each value, they generate an action (e.g. response in an experimental trial), observe the outcome and estimate the objective. They then select the location with the highest objective estimate. The circles are possible draws from the distribution when the agent just takes a single sample of the objective. Clearly these samples are quite noisy: the highest estimate is given by the right-most parameter value, even though this location is sub-optimal if the agent's objective is to maximise reward rate.

If an agent wants to gain more precise estimates of the objective, they must take several actions at each value of the strategic control parameter and integrate multiple samples of the objective at each location (e.g. compute the average objective value). As a result, the estimates will be closer to the true values (illustrated by the triangles). If the agent now picked the parameter value with the highest estimated objective value (the third point from the left), they would get quite close to the peak of the function. However, estimating the objective over a larger number of samples comes with an obvious temporal cost of generating repeated actions at each control parameter value. If time (or the total number of trials in an experiment) is limited, integrating samples over multiple actions (i.e. trials in an experiment) limits the number of different control parameter values that may be selected and the extent to which the cognitive state space may be explored.

Costs of exploring the cognitive parameter space

The temporal cost associated with sampling the objective (i.e. information gathering) creates an exploration–exploitation dilemma (Cohen et al., 2007; Sutton & Barto, 1998). That is, any time spent *exploring* a (potentially) low-value region of the cognitive parameter space is time that could have been spent *exploiting* a previously sampled location with a higher objective value—an opportunity cost. A rational agent would explore if the cost of doing so is outweighed by the long-run benefit of being able to find and exploit a region where the objective value is higher. In other words, the utility of exploration depends on the available time-horizon (Wilson et al., 2014). However, in most psychological experiments, this time horizon is relatively brief. As a result, exploration of the objective may be limited to only a small set of strategic control parameter values, rather than an exhaustive search for the peak.

In addition to this opportunity cost, sampling the objective also comes with cognitive costs. Take the simple strategy described in relation to Figure 3B: the agent samples a small number of points in the cognitive parameter space and then settles for the spot at which the objective estimate was the highest. At the very least, this agent would need to remember the location of the best objective estimate so far and the estimated value itself. They then need to update these when a better value is sampled in order to be able to return to and exploit this location once they have taken a sufficient number of samples. Moreover, they need decision mechanisms for selecting the strategic control parameters to sample and the number of samples to take. Therefore, even quite a "simplistic" sampling algorithm already requires some (working) memory resources and decision mechanisms. Note that these demands come on top of the cognitive demands imposed by the primary task itself.

The opportunity and cognitive costs associated with sampling the objective are amplified when there are multiple strategic control parameters, as in the 3D cognitive parameter space shown in Figure 2. Even if each parameter is sampled at only a few different values, their combined number scales non-linearly with the number of dimensions ("the curse of dimensionality"; Bellman, 1957). The question is how participants navigate this multidimensional space in pursuit of their objective, considering that they can only acquire noisy, local information about the objective. Addressing this question is core to the grounding of cognitive model parameters. We outline several ways in which agents might respond to this challenge.

First, as noted in the previous section, agents may choose an objective that is easier to achieve and/or estimate. For example, instead of maximising reward rate, participants may choose to maximise accuracy (Balci et al., 2011) because it may be easier to track accuracy. We have already argued for the importance of probing the objectives that are actually adopted by different agents, recognising that they may differ from the one set by the experimenter.

Second, given an objective, agents may control only one or two key parameters that have the biggest effect on this objective, thereby reducing the dimensionality of the cognitive parameter space to be searched through. For example, in an effort to maximise reward rate, participants may only modulate their decision threshold and not bother trying to control the more constrained drift rate and non-decision time parameters. As a result, they need not navigate a 3D state space (Figure 2), but only need to move along one axis (e.g. Ψ_1 in Figure 2).

Third, agents may aid search by making some prior assumptions about the objective function. They may assume that the function is smooth so that the objective value at one location is closely related to the value at a nearby location; they may assume the function is monotonic or, at least, unimodal. For instance, after sampling the two left-most points in Figure 3B, the agent might predict that stepping right is going to improve the objective value further.

Fourth, instead of searching for the parameters that yield the maximum objective value, the agent may aim for an objective value that is "good enough"—satisficing instead of maximising (Simon, 1955). All these approaches can potentially simplify the search problem, but they do not eliminate it. Even if an agent has adopted an easy-to-estimate objective, makes reasonable prior assumptions about this objective and controls only a small set of parameters in order to achieve a satisfactory value for this objective, they still need to adjust these control parameters in order to find this minimum satisfactory level. Therefore, it is paramount that any attempt to understand why participants adopted a particular set of parameters also account for the way these parameters values have been arrived at.

Sampling in cognitive parameter space

Given the constraints outlined, highly sophisticated forms of statistical optimisation (e.g. that rely on derivatives or function learning) are cognitively implausible.[4] In this section, we argue that navigating the cognitive parameter space under these constraints is a problem that is well-suited to a popular class of sampling algorithms for approximate probabilistic inference. After discussing the advantages of this framework, we will demonstrate the behaviour of an agent who adopts such an algorithm for getting around the cognitive parameter space.

In statistics and machine learning, many sampling algorithms have been developed to allow for approximate inference in complex, multi-dimensional search (or hypothesis) spaces, where inference would be intractable otherwise ("Monte Carlo" algorithms such as importance sampling, Metropolis-Hastings, Gibbs sampling, and sequential methods such as particle filtering; Andrieu et al., 2003; Doucet et al., 2001; MacKay, 2003). Consider Markov Chain Monte Carlo (MCMC) techniques that are widely used for parameter estimation in statistical and, indeed, cognitive modelling (Andrieu et al., 2003; Van Ravenzwaaij et al., 2018). Broadly speaking, these algorithms involve a set of "particles" in the (multi-dimensional) search space. Each particle represents a parameter vector for which the objective value is computed. In modelling applications, this objective is typically the likelihood or posterior density of the parameters, given some data. Each particle is then updated by randomly sampling a new *proposal* value around its current position from a transition distribution. The objective value for the proposal is compared with that of the current parameters. If the objective for the proposal is better, the proposal is *accepted* and the particle will move to that position. If the objective for the proposal is worse, the proposal is likely to be *rejected*, in which case the particle does not move; however, the proposal may still be accepted with some probability that diminishes the worse the proposal is. This cycle of generating and evaluating proposals, and updating the particles accordingly, proceeds iteratively, producing *chains* of samples. Eventually, these chains will generally be concentrated in regions where the objective value is high. However, because sampling of the objective is stochastic (by randomly sampling proposals from a transition distribution and accepting them probabilistically), lower-value regions will also be explored and the algorithm has the opportunity to escape local maxima.

In the past two decades, various authors have proposed that such sampling algorithms may also describe the way humans explore complex, open ended, internal hypothesis spaces in a wide variety of cognitive domains, such as (among others): categorisation (Sanborn et al., 2010), causal learning (Bonawitz, Denison, Gopnik et al., 2014; Bramley et al., 2017; Bramley & Xu, 2023), probabilistic inference (Dasgupta et al., 2017; Dasgupta et al., 2018), concept learning (Goodman et al., 2008; Ullman et al., 2012; Zhao et al., 2024), perception (Bill et al.,

[4] We do not claim that function learning itself is implausible; clearly it is not (e.g. Brehmer, 1976; Kalish et al., 2004; Lucas et al., 2015). Indeed, people can exploit spatial correlations in complex multi-modal, multi-dimensional functions in order to find high-value regions, with their behaviour being well described as a form of Gaussian Process regression (Wu et al., 2018). The key question here is whether these (effortful) cognitive capacities are directed to the "meta-task" of finding good strategic control parameters that govern performance in a cognitively demanding primary task.

2022; Gershman et al., 2012; Haefner et al., 2016), judgement and decision making (Lieder et al., 2018). Sampling algorithms are often presented as a cognitively plausible way for the brain to approximate rational (Bayesian) inference with lower computational overheads compared to the full normative solution (for reviews, see Bonawitz, Denison, Griffiths et al., 2014; Bramley et al., 2023; Chater et al., 2020; Fiser et al., 2010; Sanborn, 2017; Shi et al., 2010; Suchow et al., 2017; Vul et al., 2014).

Aside from their rational appeal, sampling algorithms have several features that make them attractive and plausible candidates for how an agent might navigate the cognitive parameter space under the constraints discussed previously. First, they do not require (much) prior knowledge of or assumptions about the global objective function. For sure, it is easier for these algorithms to find high-value regions when the underlying objective is smooth and parametrically simple to describe, but the algorithm does not need to "know" this in order to reach these regions. Second, MCMC algorithms search locally by generating samples randomly around the current position of the particles. Local search is consistent with the "stickiness" with which human participants update their beliefs in probabilistic inference (Dasgupta et al., 2017) or causal learning (Bramley et al., 2017). In this way, MCMC embodies a very natural trade-off between exploration and exploitation. If the algorithm has found a high-value region, samples tend to be concentrated in that region (exploitation), although there is some probability of accepting a move to a lower value region and, via that route, potentially finding a different mode (exploration). Third, the memory requirements of these algorithms are very limited. A basic MCMC sampler only needs to be able to compare a proposal sample with the current value. It does not use information from other particles or past iterations in order to extract information about the global function (of course, more sophisticated MCMC samplers do use this kind of knowledge and perform better as a result; e.g. Differential Evolution MCMC; Heathcote et al., 2019; Ter Braak & Vrugt, 2008). As such, an agent navigating cognitive parameter space may be conceptualised as a single particle (chain) moving around the state space through local search (Bramley et al., 2017; Vul et al., 2014).

In typical MCMC applications, proposals are evaluated without actually moving the particles (indeed, any move is contingent on the proposal being accepted after comparison with the current objective value). However, we suggest that human agents frequently do not have that luxury: they may not be able to simulate mentally what will happen when moving to a different location in the state space. Therefore, they can only evaluate a "proposal" location after actually moving there, interacting with the environment and getting feedback. Note the contrast with how sampling is often conceived of in other domains (e.g. judgement and decision making; Lieder et al., 2018; Stewart et al., 2006). In such scenarios, samples may be drawn mentally before committing to an overt response (even then, sampling takes time and incurs a cost; Lieder et al., 2018). Moreover, as noted earlier, a single interaction at a given location in the state space will often not be sufficient, due to uncertainty in the objective estimates. Depending on the level of uncertainty, the agent may have to remain in one place for some time and integrate feedback from several interactions (e.g. trials).[5]

Figure 4 illustrates an agent who uses a sampling approach to navigate the cognitive parameter space, looking for the peak of an objective function under the constraints discussed above. Panel A shows a 2D cognitive parameter space, with the true objective value (unknown to the agent) shown as the colour temperature for each combination of strategic control parameters (Ψ_1, Ψ_2). This particular objective function is arbitrary and chosen for illustration only. The

[5]This situation is more analogous to a modeller who can only generate model predictions through simulation and does not have access to an analytic likelihood. A standard solution in "Approximate Bayesian Computation" is to estimate a synthetic likelihood, where multiple noisy estimates of the likelihood are averaged for a given set of parameters (Hartig et al., 2011; Palestro et al., 2018).

peak of the objective is shown by the plus-sign. Panel B shows an example search trajectory of an agent who can only sample noisy estimates of the underlying objective function and who has very limited memory of their search history in this space. Specifically, at any one point in the space, the agent samples the objective function over a small window of trials and averages the estimates from that window (as described in the context of Figure 3B). If the average objective is better than that from the previously visited location, then the search continues from here; otherwise, the agent reverts to the previous location. The algorithm is described in more detail in the figure caption and in Appendix A. We assume the agent has some (coarse) information about the objective value scale and is able to perform at least an ordinal comparison between the current location and the immediately preceding one (i.e. the memory load is minimal). In the trajectory illustrated in Figure 4B the agent starts in a relatively low value location, explores some other low value regions, but ultimately ends up in a good spot, quite close to the peak objective value.

We do not claim that this algorithm is *the* algorithm that agents use to navigate the cognitive parameter space. It is entirely possible that the agent uses some other sampling algorithm or heuristic strategy. Indeed, the precise nature of the sampling algorithm should be an empirical question of interest. Sampling algorithms, like the one discussed here and others developed in studies of perception and cognition, provide a general framework for exploring this question and can accommodate a variety of strategies (Bramley et al., 2017). Our aim was to demonstrate the utility of the sampling approach as a cognitively minimal search algorithm. A secondary aim was to use this algorithm to generate trajectories in the cognitive parameter space, so that we could assess the consequences of these dynamics in the context of model-based analysis. We now turn to these consequences.

Estimating the trajectory in cognitive parameter space

At this point, it should be clear that in many psychological paradigms, there is no single point in the cognitive parameter space that is responsible for the observed data. Rather, as the agent tries to achieve an objective, they will adjust the cognitive parameters that are under strategic control, thereby moving around in this state space, as illustrated in Figure 4B. In other words, the ground-truth parameters change over time. In turn, this change produces behaviour (i.e. data) that changes over time: $\mathcal{M}(D(t)|\psi(t))$. We now switch to the perspective of the cognitive scientist who, through model-based analysis, is looking to infer the latent mechanisms that gave rise to the observed behaviour.

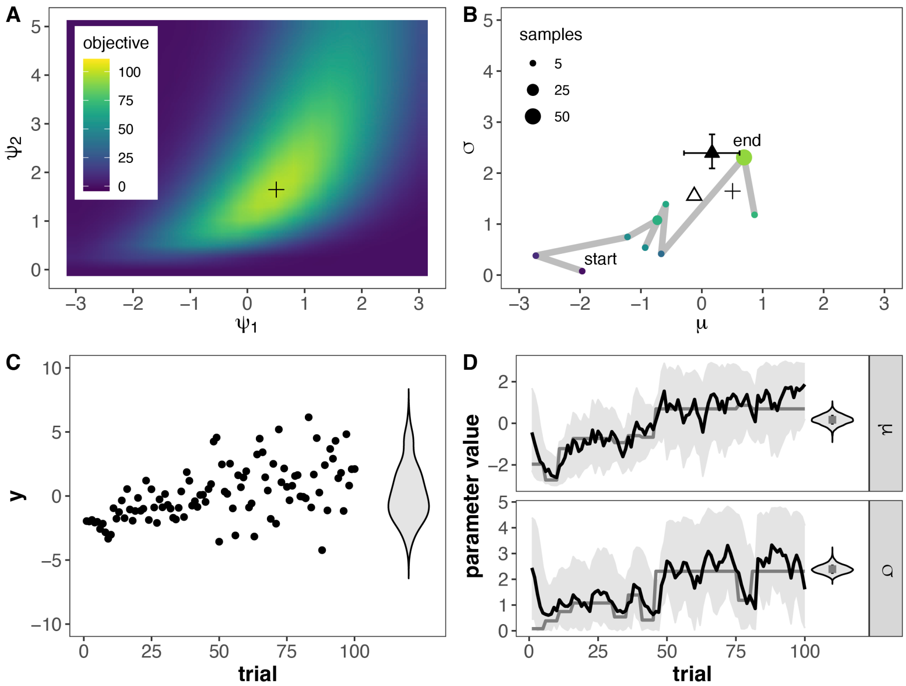
Static models are often mis-specified

For the sake of illustration, consider a cognitive parameter space defined by the mean and standard deviation of a Gaussian distribution (we deliberately frame this example in a generic, abstract format; we present a cognitively more realistic example in the next sub-section 'Capturing parameter dynamics'). Figure 4C shows a time series of data, generated by the agent's movements illustrated in panel B, with each data point $y_t \sim \mathcal{N}(\mu_t, \sigma_t)$. The agent initially generates observations from a tight distribution centred around a value of -2, but ends in a spot where they generate much more variable data with a positive mean (hence the upward trend and "fanning out" of the data). The modeller only has access to these data.

Standard practice in model-based analysis is to treat the data generating process as stationary and to estimate a single set of parameters of a cognitive model, θ, from either a set of summary statistics derived from the data or the time-unordered distribution of data. Figure 4C shows the marginal distribution of the data on the right-hand side. Suppose the modeller fits a Gaussian distribution to these data. The posterior estimates for the parameters are shown

Figure 4

Navigating a 2D cognitive parameter space with noisy objective estimates.



Note. A–An arbitrary 2D objective function. The peak of the function is marked with a '+'. The function corresponds to the mean objective value at a given location (Ψ_1, Ψ_2). An agent sampling a given location will obtain a noisy estimate of the objective around this mean. B–An example trajectory from an agent following a simple sampling algorithm over the course of N trials. For the sake of illustration, the parameter space is defined by the parameters of a Gaussian distribution. At each location, the agent generates a data point drawn from $\mathcal{N}(\mu(t), \sigma(t))$ and obtains a noisy objective estimate around the mean value shown in panel A. These estimates are averaged over a small window of W trials. If the agent jumps, the value at the new location is compared to the value at the previous location and the new location is "accepted" if the objective estimate is better. The probability of a jump, π is inversely related to the estimated objective value, $\overline{v}$, through: $\pi(\overline{v}) = 1 - \frac{\overline{v}}{\gamma}$, $0 \leq \pi \leq 1$, where γ is the threshold objective value at which the jump probability drops to 0 (ensuring search terminates when the estimated objective reaches some satisfactory value). The parameters of this simulation were: $N = 100$; $W = 5$; $\sigma_{obj} = 25$ (standard deviation around the objective); $\sigma_{jump} = 1$ (standard deviation in 2D of a bivariate Gaussian jump distribution); $\gamma = 100$. The true maximum of the objective is shown by the plus-sign. A weighted average of all the locations visited is shown by the open triangle. The filled triangle (with 95% credible intervals) shows the posterior parameter estimates based on the fit of a Gaussian distribution to the data generated by the agent. The colour temperature of the visited location indicates the true, underlying mean objective value (not the noisy estimate) on the same colour scale as the objective in panel A. C–The time series of observations generated by the agent, along with the marginal distribution. D–Posterior parameter estimates for μ and σ from a particle filter (solid black line; ribbon shows the 95% credible interval). The true parameter values are shown by the dark grey solid line underneath. The posterior densities from the static model fit are shown on the right-hand side in each panel (the posterior means, shown by the dark grey squares, correspond to the filled triangle in panel B). The vertical black line inside the densities represents the inter-quartile range (which is hard to see due to the tight posteriors).

in Figure 4D as the violin plots on the right (see Appendix B for further details). The means of these distributions are represented by the filled triangle in panel B. This simple demonstration highlights several important points. First, note that the functional form of the estimated model actually matches the ground-truth *at any one point*. However, it is clear that this model will not be a good fit when applied to the overall distribution: the data are generated by a mixture of Gaussians, with weights proportional to the number of trials spent in each location of the parameter space. As a result, the observed distribution (Figure 4C) has fatter tails, rightward skew and even a hint of bi-modality (in the right tail). In other words, exploration of the cognitive parameter space can give rise to data that appears to come from a different functional form. Second, the overall parameter estimates reflect this mixture, in that they are some kind of average of all the locations visited by the agent. However, the posterior mean or modal parameters were not actually adopted at any one point (i.e. the filled triangle in panel B does not coincide with any of the visited locations). In this case, the estimated parameters come reasonably close to the ones that were adopted most of the time by the agent (in half the trials), but of course that need not be the case if the agent explored more extensively. Third, it might be argued that Bayesian parameter estimates will reflect the temporal structure in the data implicitly in their posterior densities. That is, the posterior density of a parameter might be expected to scale with the number of trials in which that parameter was adopted by the agent. However, that appears not to be the case. For instance, the agent spends half the trials at $\mu \approx 0.7$, but the posterior density here is really low. Moreover, the point of highest density corresponds to a location where *no* time was spent. Therefore, the analyst would be hard-pressed to infer the mixture weights from the posterior densities.

Consider how the analyst might respond to this situation. We know that behaviour can change over time as participants become familiar with a task (e.g. Heathcote et al., 2000; Logan, 1992; Newell & Rosenbloom, 1981) and, therefore, we often include a (brief) practice phase in our studies. The hope (and it is often just that) is that by the time actual data collection starts, participants have settled on a point in the cognitive parameter space and their performance has stabilised. So the analyst may discard the initial data, which in this particular example would work well if they had decided (in advance) that the practice block should contain about 50 trials. However, we suspect that in many instances decisions about practice trials are not that well informed and, in any event, there is of course no guarantee that the agent will eventually settle on some stable point in the cognitive parameter space (as more or less happens in this example). Indeed, in many standard paradigms the variation in behaviour (e.g. RT) over the course of a block or the entire experiment can be much larger than the effect of any experimental manipulation(s) (e.g. Dutilh et al., 2009; Gunawan et al., 2022). So setting aside the inclusion of practice trials, and dealing with the data as they are in Figure 4C, the modeller has fit a simple Gaussian model to the observed data and notes that the fit is poor. A likely response is that they would look for a more complex functional form that can accommodate the data better, for instance an ex-Gaussian or some other skewed distribution. This model would undoubtedly produce a better fit, but is clearly mismatched to the model that actually generated the data. As a result, inferences drawn from the estimated model and its parameters are not to be trusted.

Basically, we are dealing with a form of model mis-specification: the analyst tries to approximate the dynamic ground-truth, $\mathcal{M}(D(t)|\psi(t))$, with a static model, $\mathcal{M}(\theta|D')$, where D' is a time-unordered or summary statistics representation of the actual time-series of behaviour $D(t)$. In our example, the modeller initially actually adopted the correct "basis function", but attributed the poor fit of the model to its functional form, rather than the non-stationarity of the underlying data-generating process. As a result, they were misled into adopting the wrong functional form (e.g. an ex-Gaussian). Of course, it could be argued that the modeller in this scenario should have explored their data better and realised that behaviour was non-stationary. Nevertheless, this scenario is representative of common practice in model-based analysis. For instance, evidence accumulation models are typically fit to the marginal distributions of choice and RT data over trials and modellers (including some of the present authors) often do not consider temporal structure in the data (we highlight exceptions below). Variance in the data introduced by parameter dynamics will then have to be captured somehow, for instance, through variation in the model architecture (e.g. adding noise components, such as between-trial noise in drift rate) or in the parameter values themselves (e.g. inflating existing noise components and/or pushing parameters to "compromise" values that were never adopted). In this way, a good model fit might be obtained, but at the cost of taking the modeller further away from the ground-truth.[6]

Capturing parameter dynamics

To capture the variation in the data generating process (and the resulting behavioural measurements) over time, we need to (i) consider the temporal structure in the data, the fact that the data form a *time series* of observations, and (ii) fit the time series with models that allow for temporal variation in the parameters. Of course, there are many models in cognitive science that produce or account for sequential behaviour, such as reinforcement learning models (Sutton & Barto, 1998). However, these models typically do so with a single static set of parameters; rather, the variation in predicted behaviour stems from the dynamics of and/or noise in the input (e.g. non-stationarity in the reward structure; Behrens et al., 2007). Our agenda here is to promote assessment of the dynamics in the model parameters themselves. For this purpose, it is useful to distinguish between the "core" cognitive model that generates or predicts an observation at a particular point in time, and a "meta" level of control that governs the way core model parameters change over time (for early examples of this idea in cognitive modelling, see Busemeyer & Myung, 1992; Vickers & Lee, 1998). For instance, in the example of Figure 4, the core or point-wise observation model is a simple Gaussian distribution, the parameters of which are controlled by a MCMC-like *transition model.* The question is then how to make this transition model visible (Schumacher et al., 2023).

The challenge for the analyst is to identify the sequence of hidden states from the overall collection of observations. State-space models refer to the general class of statistical techniques for solving this problem (Durbin & Koopman, 2012). Of most relevance here are flexible methods that apply to non-Gaussian and non-linear systems, such as particle filtering (Doucet et al., 2001; Gordon et al., 1993; MacKay, 2003; Speekenbrink, 2016, we review some alternative approaches in cognitive science below in the General Discussion). In particle filtering, the assumption is that an observation y_t is produced by the latent state at this time, x_t, and that the current latent state only depends on the previous state (i.e. we can define a transition model that takes us from the previous state to a new state).[7] The current state is then estimated sequentially

[6]As discussed in the previous sections and illustrated in Figures 3 and 4, parameter dynamics may stem from participants exerting strategic control (e.g. exploring the cognitive parameter space in order to achieve an objective). However, other parameters that are under less strategic control may also change with time (Dutilh et al., 2009; J. Zhang & Rowe, 2014), for example through perceptual learning (Dosher & Lu, 2017; Watanabe & Sasaki, 2015) or fatigue (Ratcliff & Van Dongen, 2011). The grounding of cognitive model parameters (and, indeed, the approaches discussed next) is not restricted to capturing only the movements due to strategic control, but can (and should) be applied to capture non-strategic changes as well.

[7]The latent state x_t may or may not include all parameters of the cognitive model we are trying to estimate. In many particle filtering applications, some of the parameters are allowed to evolve over time, whereas others remain constant (Liu & West, 2001). For our purposes, we consider the scenario where the modeller allows for movement in the parameter space along all dimensions, i.e. $x_t \equiv \theta_t$. Nevertheless, a strategy of allowing some parameters to evolve and fixing others may be a good approach for identifying the major dimensions of variation

with each incoming observation. A large population of particles represents possible latent states that generated the data. Particles associated with a high(er) likelihood for the new data are propagated, whereas those that do not capture the new data well are eliminated. In this way, the distribution of particles can adapt when the data suggest a change in the latent state. Given that we conceptualised the agent's movements in cognitive parameter space as a single particle Markov Chain, adopting a large population of particles to approximate the posterior trajectory of the agent is a natural fit.

Figure 4D illustrates the sequential parameter estimates from a basic bootstrap particle filter (Gordon et al., 1993; Speekenbrink, 2016). Details for this method are given in Appendix B. The posterior means for the parameter estimates are shown by the black solid line (with 95% credible intervals). The initial estimates are some way off, because at this point there is not yet enough data and the estimates are driven by the prior (arbitrarily centred on $\mu = 0$, $\sigma = 2.5$). However, after about 10 trials the particle filter estimates track the underlying trajectory very well, even when there are sudden larger jumps in the state space (e.g. for σ just before trial 50). The root-mean squared error (RMSE) between the mean particle filter estimates and the underlying true values is much lower than that between the static posterior mean and the true values (μ: 0.42 vs 0.89; σ: 0.46 vs 0.84).

The mixture-of-Gaussians example from Figure 4 tells a cautionary tale of the inferential pitfalls associated with fitting a static model to data from an agent who dynamically explores the cognitive parameter space. However, most computational cognitive models have more complex architectures, contain non-linearities (e.g. decision thresholds), behave non-linearly in their parameters, and may have a mixture of static and dynamic parameters. Therefore, it is important to assess whether our observations generalise to a cognitively more realistic setting. We have previously used an expanded judgement paradigm to test whether and when people collapse their decision boundaries in evidence accumulation models (Malhotra et al., 2017, 2018). In this setup, a discrete sequence of binary evidence samples points probabilistically towards one of two decision alternatives, and the agent is set the objective of maximising their reward rate. The theoretically optimal "policy" (i.e. combination of decision threshold height and gradient) may be derived for this task environment using dynamic programming (Bellman, 1957; Malhotra et al., 2018; see also Drugowitsch et al., 2012; Moran, 2015). However, the optimal dynamic programming solution is a normative strategy—it does not consider the constraints an agent actually faces when trying to adjust their decision threshold *online* as they interact with the environment and receive feedback (see also Khodadadi et al., 2014, for a theoretical analysis of this problem with regard to just threshold height). This paradigm is therefore a good test bed for generating and identifying trajectories in cognitive parameter space during a realistic (experimental) task.

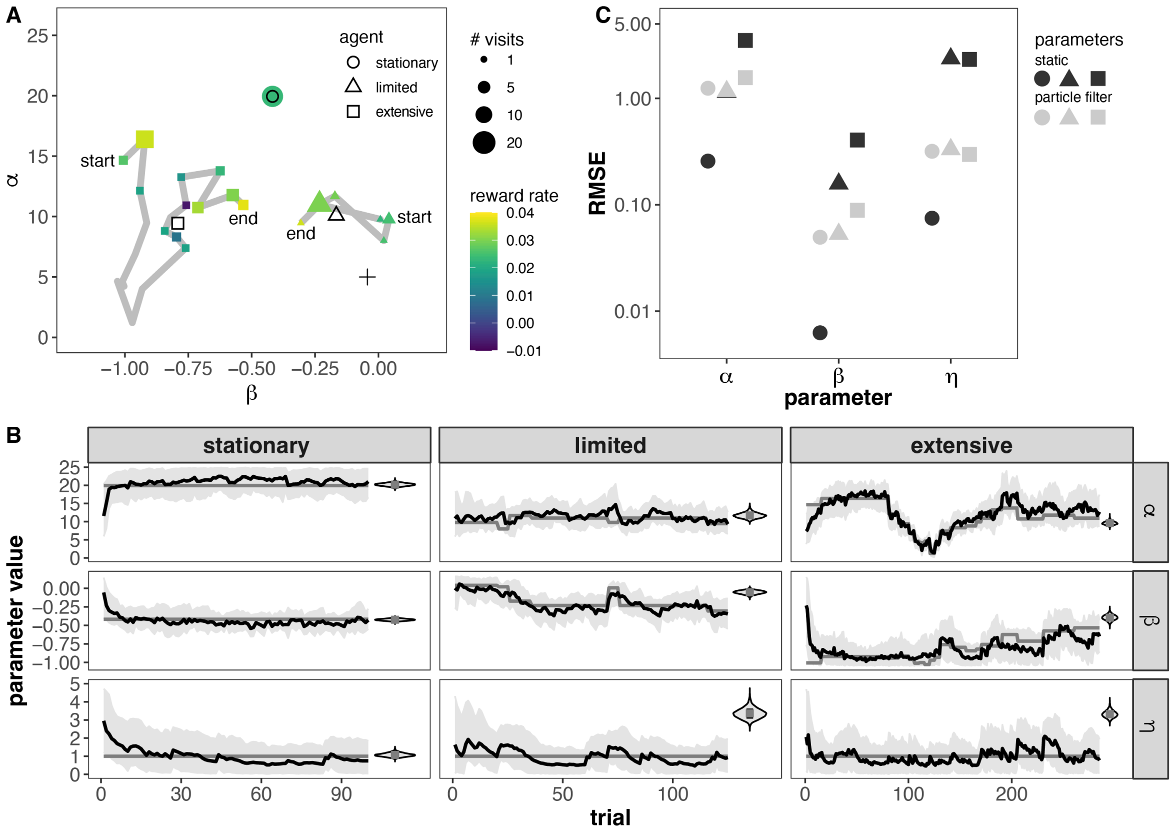
Details of the task environment and simulated agents are given in Appendix C. After each binary evidence sample, the agent makes a probabilistic decision to wait for more evidence or commit to one of the two alternatives. If they commit and their choice is correct, they are rewarded; if they make an error they are penalised. The overall amount of time is limited, so the agent will want to fit as many correct trials into this limited period as possible—they have to balance speed and accuracy. To do so, they can modulate the height and gradient of their decision boundary, after estimating the reward rate over a (small) window of trials. That is, the model assumes a 'meta' level of control that adjusts these parameters in pursuit of an objective function.

Figure 5A illustrates three different simulated agents: a 'stationary' agent and two 'non-stationary' agents who adopted the algorithm described in 'Sampling in cognitive parameter space' (see also Appendix A). These trajectories may or may not be realistic (that is an empirical

(or strategic control) over time.

Figure 5

Trajectories of three simulated agents in an expanded judgement paradigm, along with static and dynamic parameter estimates.



Note. A–Three agents who vary in the extent to which they explore the parameter space. The *stationary* agent does not move at all (by design). The two non-stationary agents adopted exactly the same sampling algorithm, but started off in different places and ended up with quite different trajectories (*limited* and *extensive* exploration). The reward rate scale corresponds to the mean reward rate averaged over all visits (although the agent only ever estimates the reward rate over a window $W = 5$ trials during the experiment). Some reward rate estimates are not shown for the extensive agent, because they were too low to be accommodated by the colour scale. The mean weighted position is shown for each agent by the open symbol. The optimal policy is indicated by the +. B–Parameter estimates derived from a static model fit (violins on the right in each panel) and from a particle filter. Conventions as in Figure 4D. C–RMSE for the different parameters and model fits. The symbol shapes correspond to the different agents, as in panel A (from left to right for each parameter: stationary - limited - extensive). Errors are shown on a log-scale to visualise the differences better (given that parameters are specified on different scales).

question we leave for future investigation). For the non-stationary agents, the "failure" to converge to the optimal policy may be down to a number of reasons: the objective function allows for a broad range of policies that generate reasonable reward rates; noise in the reward rate estimates can lead the agents astray; the sampling algorithm itself is stochastic. For the present purposes, the main thing that matters is that we have three agents who vary in the extent to which they explored the cognitive parameter space.

The core model has three parameters: threshold height α, threshold slope β, and decision noise η. Decision noise allows for a realistic amount of stochasticity in behaviour—it allows for different actions given the same amount of accumulated evidence at the same time. We fixed this parameter in our simulations to a constant value (over time and between agents), to create a scenario in which the ground-truth model contained a mix of dynamic and static parameters. Of course, the modeller does not know a priori which parameters evolve and which ones do not. We suggest this situation is a common occurrence in many cognitive modelling applications. Figure 5B illustrates the estimates for each of the three model parameters. On the right-hand side of each panel, the violin plots show the posterior densities of the parameters estimated from a static model (i.e. a single set of $\{\alpha, \beta, \eta\}$ for the entire run of trials). When the agent is stationary, these estimates are highly accurate and precise. For the non-stationary agents, as in our earlier example (Figure 4D), the static estimates to some extent reflect a weighted average of the threshold parameters that were adopted by the agent. However, the noise parameter is greatly over-estimated. The reason for this error is straightforward: movement in the latent state space introduces variability in behaviour. The static model can only capture this variability by increasing the noise component.

Now consider the estimates of the particle filter. For the stationary agent, the particle filter estimates are concentrated around the true values, albeit with much greater variance compared to the static estimates. The non-stationary agents are tracked very well with the particle filter. There are some periods where there seems to be some parameter trade-off (e.g. for the extensive agent towards the end, higher intercepts are compensated for by steeper gradient). However, similar trade-offs can occur for the static fits (e.g. for the extensive agent, the overall intercept estimate seems to be underestimated and the gradient overestimated). More importantly, the particle filter adequately captures the level and constancy of the noise. Figure 5C summarises the results by plotting the RMSE for the three agents and parameters. Unsurprisingly, the advantage of the particle filter is more pronounced the greater the movement in the state space (most clearly illustrated for parameter α; note the logarithmic scale which visually compresses the differences at the higher end). Overall, there appears to be an asymmetry in the consequences of model mis-specification: when the data are generated by a stationary agent, the cost of allowing parameters to vary over time is not great (although the estimates are clearly much less precise); when the data are generated by a non-stationary agent, the cost of a static approximation can be severe. In summary, allowing for parameter dynamics not only enables identification of the agent's journey in cognitive parameter space (or the near absence of one), it also guards against erroneous inferences that can arise when a static model has to capture variability in behaviour that stems from non-stationary latent states.

General Discussion

Computational cognitive modelling is increasingly ubiquitous in psychology, neuroscience and psychiatry. A model-based analysis of behaviour is used to provide mechanistic explanations of empirical phenomena such as experimental effects, neural activation and individual or population differences. This type of analysis can generate important insights, such as those listed in Table 1. However, we have argued that more complete explanations should also address *how* and *why* model parameters take on the values that they do. To address these questions, we need to consider the cognitive parameter space—a latent state space formed by the parameters of the (ground-truth) model that generates the data—from the perspective of the agent *while they are engaged in a task.* The agent may be trying to achieve some objective (either one they have formulated themselves or one that is imposed by the experimenter). They are unlikely to have a good representation of how their position within the (internal) cognitive parameter space maps to the (external) objective value. This is a non-trivial problem for the agent to solve, firstly because the information they obtain about the objective is probably highly uncertain. Secondly, their room for manoeuvre in the state space is limited by various biophysical and cognitive constraints. Under these circumstances, the agent may be feeling their way around the task, trying to find a region in the state space where they meet their objective. We have set out a programme of *grounding cognitive model parameters* that involves a mix of empirical and computational work to identify the constraints that the agent operates under, the objective(s) they adopt, the information available to them to estimate their objective online over the course of task performance, and the mechanisms by which they move around the state space to achieve their objective.

Part of this endeavour involves an assessment of the agent's behaviour from one moment (trial) to the next, taking into account the feedback they receive from the environment. This perspective is often missing from a model-based analysis, where a single set of parameters is typically estimated on the basis of a set of summary statistics or time-unordered representation of the data. At best these estimated parameters provide a good reflection of the average position of the agent over the course of the task. However, in most cognitive models the relation between parameter values and predicted behaviour is often highly non-linear, and there is no guarantee that the average position of the agent may be recovered from the time-unordered data. Moreover, the movement of the agent in the state space introduces variability in behaviour over time that will have to be absorbed somehow in the static model parameters. As our simulations show (Figures 4 and 5), this can lead to poor model fits, seriously biased parameter estimates and inferential errors. A poor fit might lead the modeller to consider alternative static models (with additional mechanisms or more complex functional form) that may do a better job in capturing the average behaviour, but that nevertheless take the analyst further away from the non-stationary ground-truth.

Related approaches

We are of course by no means the first to discuss these topics. Each of the sections addressing the four topics listed above, contains numerous recent citations, suggesting that a good deal of relevant work has been and is being conducted. For instance, there is burgeoning interest in capturing the dynamics of behaviour by modelling the time-dependence of parameters of either descriptive, statistical or cognitive models (e.g. Gunawan et al., 2022; Kunkel et al., 2021; Miletić et al., 2021; Schumacher et al., 2023). For example, Hidden Markov Models may be used to identify a small set of latent states that gave rise to the observed data—with participants switching between a limited number of cognitive "regimes" (e.g. on/off-task; cf. Gunawan et al., 2022; Kunkel et al., 2021; Visser, 2011). Alternatively, when parameters change smoothly over time (e.g. as a result of practice or fatigue), the temporal trend may be captured by some simple parametric form (e.g. polynomial; Gunawan et al., 2022). The challenge is then to estimate the parameters of this functional form that, in turn, controls the core model parameters at any one point in time. Part of our contribution then is to place recent work like this in the context of a wider research programme of grounding cognitive model parameters and open up avenues for future research. In that regard, estimating these trajectories is just a starting point—they still need to be related to the substantive psychological questions about the constraints, objectives and mechanisms involved in navigating cognitive parameter space.

In our simulations, we adopted a particle filter approach for several reasons. First, most computational cognitive models are highly non-linear and non-Gaussian, which is the domain particle filters are designed for. Second, particle filtering is an inherently flexible approach, so that we need not commit to a fixed number of latent states in advance (as in Hidden Markov Models) or to a particular fixed functional form of trajectory through the state space (e.g.

polynomial). Third, and most importantly, we can think of the agent as a single particle MCMC chain in cognitive parameter space. Using a population of particles to estimate their trajectory therefore seems like a natural choice.

Nevertheless, ideally we would include the *cognitive mechanisms* that underlie the trajectory in the state space into our models directly. One example of this approach is to have a subset of parameters controlled by some learning algorithm that responds to feedback in the environment and updates parameters accordingly (e.g. drift rates or starting points in evidence accumulation models; Fontanesi et al., 2019; Ludwig et al., 2012; Miletić et al., 2020; Miletić et al., 2021; Pedersen et al., 2017). Similarly, it may be possible to extend our cognitive model with sampling-like mechanisms for exploring the cognitive parameter space. For instance, the local sampling and comparison routine we outlined as a cognitively minimal search algorithm, is characterised by a number of parameters (e.g. the scale of the jump distribution, the window of trials over which the objective is estimated). Recently, Schumacher et al. (2023) developed a general framework for estimating the parameters of a transition model along with the core model parameters that determine the response at a given moment in time. This framework is applicable to a wide variety of possible transition models, and it is conceivable that we might specify more cognitively plausible transition models (and test between them). We see this challenge as an important and exciting problem to be addressed in future work.

There are also long-standing efforts to ground model parameters through a *rational* or *computational* analysis of cognitive capacities and the world they operate in (Anderson, 1990; Lieder & Griffiths, 2020; Oaksford & Chater, 2007).Some of our arguments echo these approaches. Specifically, a computational or rational analysis requires that we consider the task objective and the information available in the environment for the agent to pursue that objective. Importantly, there is a strong normative commitment to adaptive objectives and the rational use of information—a focus on what the agent *should* do given the task, the environment in which they operate and, potentially, their cognitive constraints (Anderson, 1990; Lieder & Griffiths, 2020). Our analysis encourages researchers to recognise the variety of objectives that might be adopted, to investigate the *actual* objectives that were adopted, and to model the cognitive mechanisms by which these (variable) objectives were pursued (see also Rahnev & Denison, 2018, for similar arguments in the specific domain of perceptual decision-making). The ability of humans to make up their own objectives, and the mechanisms by which they pursue those objectives, are fundamental to their psychological make-up. Endowing cognitive models with the capacity to generate this variability is a key step towards a better understanding of variation in cognition and behaviour between different individuals and over time. The rational analysis of a cognitive capacity can certainly provide useful guidance in this enterprise and it may be a good place to start.

Challenges and pragmatic considerations

Grounding computational cognitive models involves a form of "zooming out": going beyond the core model and interrogating the systems that provide input to this core and/or control the way the core model adapts over time (e.g. to achieve some objective). Zooming out is likely to expand the model with additional mechanisms (e.g. a perceptual front-end; sampling mechanisms), a "meta-level" of input and control. Such expansion raises a number of questions, obstacles and objections. First, how far do we take this expansion? Second, additional mechanisms are likely to make models more flexible and more difficult to falsify. Third, it can already be difficult to estimate parameters accurately and reliably for the core model, and building more complex models will make it even harder.

If we expand a core model with higher-level mechanisms that control the ones lower down, those control mechanisms themselves may have parameters in need of explanation. How many layers of control do we build? And how do we stop model complexity spiralling out of control? In our view the correct level of explanation should be determined by the questions that a researcher wishes to answer. Some meta-level model will be required to answer questions about fine-grained behaviour, such as how participants navigate cognitive parameter space in order to adapt behaviour to meet some objective. It is entirely conceivable that interesting questions can be asked about the mechanisms at this meta-level, such as the generality of the transition model across different task domains, the flexible tuning of its parameters (e.g. scale of the jump distribution), and so forth. We certainly do not advocate an extensive regress of meta-meta-…-level models and complexity for complexity's sake. At some (meta-meta-…-)level, the higher level questions may no longer be interesting, or we may simply lack sufficiently diagnostic data that speak to these questions. However, we hope to have persuaded the reader that there is great scope for expanding model-based analyses to incorporate at least one layer of meta-level processes.

Nevertheless, it is undeniable that model expansion brings with it major challenges in parameter estimation and model selection. For instance, there is the bias-variance trade-off, clearly visible in Figure 5B. When the ground-truth is non-stationary, estimating a single, static set of parameters yields more precise estimates: the posterior densities tend to be narrower than the 95% credible intervals around the time-varying estimates. However, the static parameter estimates are precise, but wrong: they can be a poor reflection of the system at any one point in time and/or display a large amount of bias (e.g. inflated noise estimates). The time-varying estimates track the temporal evolution in the latent states, so that this variation is not accommodated through biasing other model parameters. When the ground-truth is stationary, the difference in estimation precision is even more pronounced, although in this case both the simple static and the time-varying estimates show little bias. What level of bias and variance is acceptable depends on the modelling goals. For instance, bias may be acceptable if the primary aim is the detection of some experimental effects and the model is simply a vehicle for extracting a more sensitive measure from multi-dimensional data (van Ravenzwaaij et al., 2017). However, when the aim is to understand the underlying mechanisms that generated the data better, the loss in estimation precision may be a price worth paying for avoiding bias and gaining insight into the way participants move through the state space.

If we consider movement in cognitive parameter space seriously, it is clear that a wide variety of trajectories are possible (e.g. the two non-stationary agents in Figure 5A). It is likely that participants will start off at different points in the state space, as a result of their own motivations, background knowledge, history with broadly similar tasks, and their own conceptualisation of what the task is (Szollosi et al., 2023). Once you take into account the possibility of different transition models, noise in the objective estimates, and stochasticity in the transition model, it will be extremely difficult to disentangle these many sources of variability. There is no single approach to deal with these challenges, but our paper suggests a number of possible empirical and computational avenues to explore (e.g. setting clear task objectives, identifying objectives that were actually adopted). At least at a practical level, there are various tools for estimating individual trajectories, as demonstrated by our own simulations (Figures 4 and 5), and by other authors recently (Gunawan et al., 2022; Schumacher et al., 2023). The estimated trajectories may then become the data of interest for the development and tuning of more cognitively principled search algorithms. For instance, if an estimated trajectory suggests that a participant tends to step in the same direction after a previous move improved their objective value, that behaviour would suggest some form of hill-climbing (Bramley et al., 2017; Busemeyer & Myung, 1992); however, if they took very large steps and behaviour was uncorrelated from one epoch to the next, that might suggest a much more random search process, such as independent sampling. In any event, just because identifying individual trajectories in cognitive parameter space is a formidable challenge, pretending that they do not exist is likely to lead us down inferential blind alleys.

Finally, trying to understand the trajectories in cognitive parameter space implies a shift of focus in the development of our theories and models. Current model-based practice typically aims to capture some average behaviour over the course of a single, specific task. We invite the modeller to take the view of the participant navigating that task, from one moment to the next. This perspective entails a focus on the mechanisms by which cognition and behaviour evolve over time and adapt to a set of task demands (Bramley et al., 2023; Donkin et al., 2022). Typically, participants are feeling their way through a task, with little knowledge of how their position in the cognitive state space influences the objective they are looking to achieve. As a result, different participants will generate very different trajectories in the state space and produce very different (average) behaviours. Returning to the two agents illustrated in Figure 5A, the static parameter estimates suggest very different strategies (policies) for solving the task. At the same time, at the higher control level, the agents' strategies *were exactly the same* (by design in this instance). They had exactly the same objective and used exactly the same sampling algorithm to explore that objective. As cognitive scientists, we are often interested in robust invariances that generalise across environments, tasks and people. Perhaps it is at the level of these more general mechanisms that many interesting cognitive invariances will be found.

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Appendix A

Appendix A – Local sampling and comparison algorithm

Here we provide a more detailed description of the local sampling and comparison algorithm used to generate trajectories in the cognitive parameter space (Figures 4 and 5 in the main text). The objective function used for Figure 4 is arbitrary and defined as:

$$f(\Psi_1, \Psi_2) =$$
$$A \exp\left(-\frac{1}{2(1-\rho^2)} \times \frac{(\Psi_1 - \mu_{\Psi_1})^2}{\sigma^2_{\Psi_1}} - 2\rho \times \frac{(\Psi_1 - \mu_{\Psi_1})(\ln \Psi_2 - \mu_{\Psi_2})}{\sigma_{\Psi_1}\sigma_{\Psi_2}} + \frac{(\ln \Psi_2 - \mu_{\Psi_2})^2}{\sigma^2_{\Psi_2}}\right),$$

with the following constants: $A = 100$; $\mu_{\Psi_1} = \mu_{\Psi_2} = 0.5$; $\sigma_{\Psi_1} = 1$; $\sigma_{\Psi_2} = 1.5$; $\rho = 0.75$. The objective function for the expanded judgement paradigm simulated for Figure 5, is determined by the parameters of the behavioural task, given below in Appendix C.

Agents are initialised at random points in the cognitive parameter space. At any one point, the agent interacts with the environment and estimates the objective value V with some uncertainty, i.e. $p(V = v)$. To reduce the uncertainty, the agent integrates the objective estimates over a window of W trials, to obtain $\overline{v}$. With the total number of trials, N, being limited, the sampling window should be small enough to allow for some exploration, i.e. $W << N$.

If the estimated objective value, $\overline{v}$, after W trials is low, the agent is likely to jump to a different location in the space. If the value is high, the agent is likely to stay in the current location and perform another set of W trials. Specifically, the probability of a jump, π depends on the estimated objective value through: $\pi(\overline{v}) = 1 - \frac{\overline{v}}{\gamma}$, $0 \leq \pi \leq 1$, where γ is the threshold objective value at which search can be stopped. The precise form of this function is not too important and the agent does not need precise knowledge of the maximum attainable objective value. What matters is that they have some crude sense of whether a location is good or bad, and that this estimate bears an ordinal relation with the true objective value. Whether the agent jumps is a Bernoulli random sample with $p = \pi(\overline{v})$.

If the agent jumps, a "proposal" location is drawn from a symmetrical transition distribution around their current location; in other words, the jump is essentially random, but likely to be close by. We formalised the transition distribution as a multivariate Gaussian, but again the precise implementation is probably not too important. We could have chosen a Lévy flight, a uniform distribution or some other form; the scale of the distribution is much more important, so that the jumps are of an appropriate size. After a transition, the agent samples the objective for W trials at the new location. If the estimated objective is better than the estimate from the previous location, the proposal location is "accepted" and search continues from here. If the estimated objective at the new location is worse than the estimate from the previous location, the proposal location is "rejected" and the agent reverts to the previous location. After an accept/reject decision, search continues as before: jumping with a probability that depends inversely on the estimated objective value at the new location (if the proposal was accepted) or the previous location (if the proposal was rejected).

One way an agent may simplify the search process is through satisficing. This behaviour is controlled by the stopping criterion γ, which controls when the jump probability goes down to 0. Satisficing then involves setting γ to a value below the peak of the objective function. Again, the precise implementation details do not matter a great deal; the key ingredient is that agents have some way of sensing that a particular objective value is good enough and that this sensation drives down the probability of jumping. In our simulations, γ was always set to the true maximum of the objective (i.e. 100 for the arbitrary objective from Figure 4A; the peak reward rate derived from the dynamic programming solution in Figure 5). In other words, our agents were maximisers. Note that as a result of noise in the objective estimates, it can (quite easily) happen that the estimated value at a given location exceeds the stopping criterion. In this case, the jump probability is set to 0 and the agent does not move. However, the next window of trials at the same location might easily generate an objective estimate lower than γ, in which case there is some non-zero probability of a transition. Therefore, noise in the objective values may result in the agent moving away from a good (or even optimal) spot.

For the example trajectory shown in Figure 4B, the agent arrives at a good spot on the surface and is able to spend just over half their time there. However, this outcome is not guaranteed: every run of the algorithm will produce a different result, as demonstrated in Figure 5 (top row). The performance of the algorithm will depend on its parameters (W, σ_{jump}, γ), the nature of the objective surface and the amount of noise in the objective estimates. Variation in some or all of these components will produce variation in behaviour over time (as search progresses) and variation between individuals.

The algorithm described here is similar to standard Markov Chain Monte Carlo sampling algorithms such as Metropolis-Hastings, except that: (i) it operates on estimates derived from several (i.e. W) trials rather than just a single evaluation of the objective (target); (ii) a proposal is generated and evaluated with a (jump) probability that is inversely related to the estimated height of the objective; (iii) the agent simply rejects proposals with a lower estimated objective value than the previous position (rather than accepts them with some probability). The first modification ensures that the agent has a more reliable estimate of the objective (analogous to the synthetic likelihood approach in Approximate Bayesian Computation; Hartig et al., 2011; Palestro et al., 2018). The second modification is necessary, because we assume that the agent actually has to move to a new location and interact with the environment in order to gain information about the objective at that location (i.e. they cannot estimate the objective elsewhere through, say, mental simulation). Such a move is potentially costly, because the new location may have a lower objective value and the agent will have to spend W trials there to find out. Therefore, rather than always generating proposals (as in standard MCMC), we want to do so adaptively, depending on how good or bad the current location is. The third modification is appropriate in this setting, because the agent's goal is not to sample the full objective function with a probability that is proportional to its value—the goal is simply to find the best (or good enough) location and spend as much time there as possible.

Appendix B

Appendix B – Static and dynamic parameter estimates for mixture-of-Gaussians example

For the simple model used to generate Figure 4 in the main text, we used the following procedures to estimate parameters.

Static parameter estimates

For the static model, we simply need to estimate the mean and standard deviation for a Gaussian distribution, i.e. $y \sim \mathcal{N}(\mu, \sigma)$. We adopted the following priors for the parameters μ and σ:

$$\mu \sim \mathcal{N}(0, 1)$$
$$\sigma^2 \sim \text{inv-}\chi^2(5, 5)$$

We approximated the posteriors through sampling, using the Stan probabilistic programming language (Stan Development Team, 2024b) and the RStan interface (Stan Development Team, 2024a). Parameters were bound to the intervals $\mu \in [-3, 3]$ and $\sigma \in [1\text{e-}6, 5]$. We used four chains with 1k post-warm-up iterations, giving 4k posterior samples for each parameter.

Particle filter

Latent states correspond to the time-evolving parameters of the model: for each trial, we assume there is an underlying state $\theta_t = \{\mu_t, \sigma_t\}$. We implemented a basic bootstrap particle filter to approximate the posterior distribution of latent states at each time point (Durbin & Koopman, 2012; Gordon et al., 1993; Speekenbrink, 2016). We used $J = 2000$ particles. The initial locations of the particles are drawn from a bivariate Gaussian distribution with prior mean $\mu_0 = \{0, 2.5\}$ and unit variances $\Sigma_0 = \mathbb{I}$, where $\mathbb{I}$ denotes the identity matrix. Particles are assigned weights of $w_t^{(j)} = J^{-1}$. For each incoming observation y_t, with $t = 1, \ldots, T$, the distribution of particles then evolves as follows:

- Propagate particles through a *transition distribution*, $p(\theta_t^{(j)} | \theta_{t-1}^{(j)})$. We chose a bivariate Gaussian with $\Sigma_{trans} = \text{diag}(0.5, 0.5)$.
- For $j = 1, \ldots, J$, compute the likelihood of the observation, i.e. $p(y_t | \theta_t^{(j)})$.
- Weight update: $\tilde{w}_t^{(j)} = w_{t-1}^{(j)} p(y_t | \theta_t^{(j)})$. This step assigns higher weights to particles that are more consistent with the new observation. Normalise the weights: $w_t^{(j)} = \frac{\tilde{w}_t^{(j)}}{\sum_{j=1}^{J} \tilde{w}_t^{(j)}}$.
- To avoid weight degeneracy, resample the weights if they are concentrated on too few particles, i.e. if $\frac{1}{\sum_{j=1}^{J} (w_t^{(j)})^2} < 0.5J$. Resampling eliminates particles with low weights and replicates particles with higher weights. If the resampling step was performed, set $w_t^{(j)} = J^{-1}$.

Particles were constrained to the intervals $\mu_t \in [-3, 3]$ and $\sigma_t \in [1\text{e-}6, 5]$. At each iteration, the collection of particles and their associated weights may be used to compute various (weighted) quantities of interest. In Figure 4D, we show the means, along with the 2.5^{th} and 97.5^{th} percentiles, of the marginal posteriors.

Appendix C

Appendix C – Expanded judgement task simulations

Task environment

The agent is presented with a sequence of binary evidence samples, $\mathbf{x}$ with $x_i \in \{-1, 1\}$. That is, each evidence sample points to one or the other decision alternative, with a consistent bias during any one trial, i.e. $x_i \sim \text{Bernouilli}(0.5 \pm \epsilon)$. The agent terminates the evidence stream by committing to a decision alternative. The accuracy of that decision determines whether they are rewarded or penalised. The next trial then starts after a delay. If there is time remaining in the task, the next trial is then presented. For the agents simulated in the current paper, we adopted the task parameters listed in Table C1.

Table C1

Parameters of the simulated expanded judgement task

Parameter	Value
Inter-stimulus interval (ISI)	0.2 s
Inter-trial interval (correct)	3 s
Inter-trial interval (error)	3 s
Monetary reward (correct)	1
Monetary penalty (error)	-2
Evidence bias, ϵ	0.2
Within-trial response deadline	10 s
Task duration	480 s

Decision model

Malhotra et al. (2018) described the optimal decision policy for this task, derived through dynamic programming (Bellman, 1957). This policy prescribes what the agent should do for each possible combination of time (number of evidence samples observed) and accumulated evidence. A given decision policy may be represented as a tripartite division of the (time, evidence)-space: a region where the agent should go for option A+, a region where the agent should go for option A-, and a region in between where the agent should wait for further evidence. The boundaries separating the 'go' from the 'wait' regions are symmetric and may be summarised conveniently by an intercept and slope. Let α and β represent, respectively, the intercept (in evidence units) and gradient (in radians) of the positive decision boundary. The long-run expected reward rate can then be computed for each (α, β) combination (Malhotra et al., 2017). This 2D objective function, given the current task parameters, is illustrated in Figure C1A.

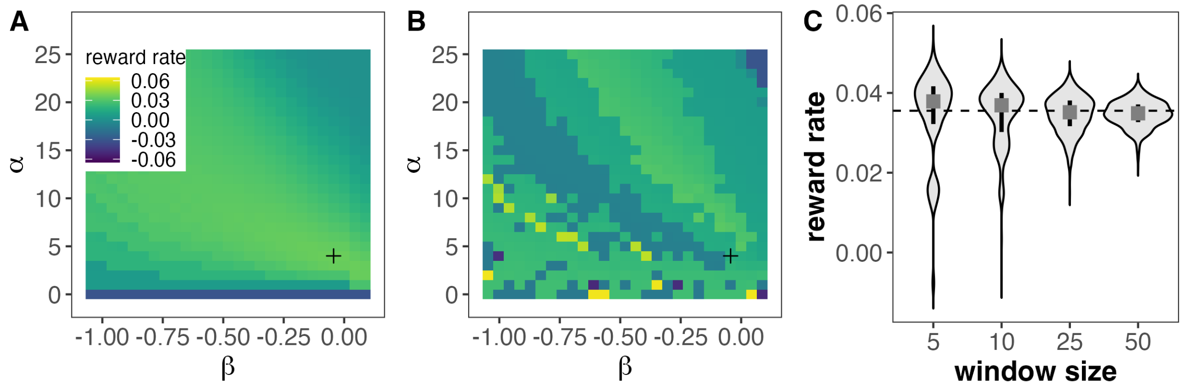
Our simulated agents essentially adopt this model for making their decisions. On a particular trial t, their (positive) evidence boundary at the k^{th} sample is given by: $b^+(k) = \max(\alpha_t + k \tan \beta_t, 0)$; the negative boundary is simply $b^-(k) = -b^+(k)$. Within a trial the agent integrates the evidence perfectly, so that the decision variable after k samples is: $z_k = \sum_{i=1}^{k} x_i$. However, to introduce a realistic amount of variability in behaviour we assume that the agent makes a probabilistic 'wait'/'go' decision after each sample. That is, they draw a sample from a Gaussian distribution centred on z_k: $z^* \sim \mathcal{N}(z_k, \eta)$, where η corresponds to the decision noise. If $z^* \geq b^+(k)$, the agent goes for option A+; if $z^* \leq b^-(k)$, the agent goes for option A-; if $b^+(k) = b^-(k) = 0$, a random choice is made between A+ and A-; otherwise, the agent waits for the next evidence sample. The decision noise parameter generates variability in choice and decision times—it ensures that the agent does not always make the same choice when presented with the same accumulated evidence at the same point in time. This simple decision model is characterised by three parameters: $\theta = \{\alpha, \beta, \eta\}$.

Navigating cognitive parameter space

Our simulated agents have two parameters under strategic control: the height and gradient of their decision boundary. Decision noise is constant and set to $\eta = 1$ throughout. Figure C1A shows the long-run expected objective values across this 2D cognitive parameter space for an agent without any decision noise. There is a broad ridge in this space where good reward rates may be obtained: higher boundaries (which would slow people down) may be compensated for by having them collapse more steeply (which forces a decision by effectively imposing a deadline; Hawkins & Heathcote, 2021). Each agent is initialised in a random position within the cognitive parameter space, with $\alpha \in [0, 25]$ and $\beta \in [-1.047, 0.087]$ radians. However, they *never really* experience the long-run expected reward rate for a given threshold height and gradient. Aside from the particular decision policy adopted, the experienced reward rate depends strongly on the stochastic evidence sequences that are presented, the amount of decision noise and the window over which the agent estimates the reward rate.

Figure C1

Reward rates in a simulated expanded judgement paradigm



Note. A–Reward rate from dynamic programming. B–Noisy reward rates estimated over a window of 5 simulated trials. At each point in the state space, agents were presented with the same (initial) sequence of evidence samples. Due to differences in the policies these samples will be distributed differently across trials. For instance, agents who sample more (e.g. due to a high intercept) will see evidence samples that were never seen by agents who sampled much less (e.g. due to a low intercept). C–Distribution of reward rates over 1000 windows of varying sizes.

Figure C1B and C give an indication of the variability in reward rate an agent might actually experience when estimating the reward rate from using a window of W trials. Panel B shows the experienced reward rate across the cognitive parameter space with $W = 5$ trials. Clearly, the globally "true" optimal policy (marked with a $+$) is not necessarily optimal for a particular window of trials. Panel C shows the distribution of reward rates experienced for repeated windows of 5 trials at the same location (in this case, the globally optimal location). The long-run expected reward rate is indicated by the horizontal dashed line. We show these distributions for a range of window sizes. As expected, larger windows yield estimated reward rates that are more tightly clustered around the long-run expected value. However, these simulated agents generally performed between 100–300 trials over the course of the experiment (cf. Figure 5), so a window of 50 trials would not allow for much exploration. The variability in experienced reward rate (particularly for smaller, more realistic window sizes) is very pronounced and will make it extremely difficult to find the optimal policy in a limited duration (Evans & Brown, 2017; Malhotra et al., 2017).

We are not so much concerned with whether agents have sufficient information to find the optimal position in the cognitive parameter space. Rather, our aim was simply to generate trajectories in the state space and assess the resulting parameter estimates. To generate the trajectories for the non-stationary agents shown in Figure 5, we followed the local sampling and comparison routine described in Appendix A. Specifically, we adopted the following parameters for the algorithm: $W = 5$; the standard deviation of the jump distribution was set to 10% of the permissible range of the parameters, i.e. $\Sigma_{jump} = \text{diag}(2.5, 0.113)$ (for α and β respectively; note these are diagonal variances); $\gamma = 0.036$ (close to the true optimal reward rate). The stationary agent was given a random (initial) position, just like the non-stationary agents, but was simply not allowed to move.

Parameter estimation

Static parameter estimates

Given the decision model described above, and given a particular decision boundary and accumulated evidence after sample k, the probability for the three possible actions $a \in \{\text{A+}, \text{A-}, \text{wait}\}$ are:

$$p(a = \text{A+}) = 1 - \Phi(b^{+}(k), z_k, \eta)$$
$$p(a = \text{A-}) = \Phi(b^{-}(k), z_k, \eta)$$
$$p(a = \text{wait}) = 1 - p(a = \text{A+}) - p(a = \text{A-}),$$

where Φ denotes the cumulative Gaussian distribution function. The likelihood of an observed sequence of K actions on trial t is now simply: $\mathcal{L}_t = \prod_{k=1}^{K_t} p(a = a_k^* | \theta)$, where a_k^* indicates the chosen action after sample k. Across the entire set of trials $t = 1, \ldots, T$, likelihoods may combined through: $\mathcal{L} = \prod_{t=1}^{T} \mathcal{L}_t$. As usual, it is more convenient to compute the log-likelihood: $\ln \mathcal{L} = \sum_{t=1}^{T} \sum_{k=1}^{K_t} \ln p(a = a_k^* | \theta)$.

We adopted the following priors and parameter transformations:

$$\alpha^*, \beta^*, \eta^* \sim \mathcal{N}(0, 1)$$
$$\alpha = \exp(\alpha^* + 1.5)$$
$$\beta = 0.5\beta^*$$
$$\eta = \exp(\eta^* + 1).$$

For α and η, the prior and subsequent transformation result in right-skewed distributions, bound at 0 from below and with most of their mass between $0 - 10$ (with a longer tail for α). For β, the prior is Gaussian and centred on a gradient of 0, i.e. a flat decision boundary. Posterior densities for the parameters were again approximated using Stan, using four chains with 5k post-warm-up iterations. After thinning the chains by a factor of 5, we were left with 4k posterior samples for each parameter.

Particle filter

We adopted the perspective of a modeller who does not know which parameters change over time and which ones do not. So all parameters were allowed to evolve: $\theta_t = \{\alpha_t, \beta_t, \eta_t\}$. We used the bootstrap particle filter described in Appendix B, again with $J = 2000$ particles. Initial particle locations were drawn from a multivariate Gaussian prior with $\mu_0 = \{5, 0, 2\}$ and variances $\Sigma_0 = \text{diag}(25, 0.16, 9)$. The prior needs to be wide enough so that there will be some particles that account for the early data. The transition distribution was also a multivariate Gaussian, with variances $\Sigma_{trans} = \text{diag}(1, 0.0025, 0.04)$. The scale of this transition distribution determines how quickly particles can adapt when there has been a change in the the latent state (i.e. if particles can only move a small amount, it will take a long time to adapt to a large jump in the state space). Parameters were constrained to the intervals $\alpha_t \in [0.01, 25]$, $\beta_t \in [-1.05, 0.175]$, and $\eta_t \in [0.01, 5]$.